

On the deformation theory of $\mathbb{E}_\infty$ -coalgebras

Florian Riedel

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Abstract

We show that, for any prime p and any space X , the $\mathbb{F}_p$ -chains $C_*(X; \mathbb{F}_p)$ admit a unique and functorial lift to an $\mathbb{E}_\infty$ -coalgebra in the ∞ -category of p -complete connective spectra, given by the p -completed suspension spectrum $(\Sigma_+^\infty X)_p^\wedge$. We do this by introducing a notion of formally étale coalgebras and prove that the coalgebraic right adjoint to base change along the map $\mathbb{S}_p^\wedge \rightarrow \mathbb{F}_p$ is fully faithful when restricted to formally étale coalgebras. We conclude by showing that the $\mathbb{F}_p$ -chains $C_*(X; \mathbb{F}_p)$ are formally étale, by reducing to the case of an Eilenberg–MacLane space $X = K(\mathbb{F}_p, n)$ and using an explicit presentation of the coalgebra $C_*(K(\mathbb{F}_p, n); \mathbb{F}_p)$ via power operations.

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1 Introduction

Overview

Let X be a space and throughout fix a prime p . We want to compare two different invariants associated to X , namely the $\mathbb{F}_p$ -chains $\mathbb{F}_p[X]$ and the p -completed suspension spectrum $\Sigma_+^\infty X_p^\wedge = \mathbb{S}[X]_p^\wedge$. The former can be recovered from the latter as the base change

$$\mathbb{S}[X]_p^\wedge \otimes \mathbb{F}_p \simeq \mathbb{F}_p[X],$$

but on the level of spectra, there is generally no canonical way to go back.¹

However, we can rigidify this situation by considering additional structure on the chains. Namely, for any $\mathbb{E}_\infty$ -ring R , the R -valued chains $R[X]$ come equipped with the structure of an $\mathbb{E}_\infty$ -coalgebra, induced by the diagonal map $\Delta: X \rightarrow X \times X$. Our main result is that, as an $\mathbb{E}_\infty$ -coalgebra, the p -completed suspension spectrum is completely determined by the $\mathbb{F}_p$ -chains:

Theorem A. For any $X \in \mathcal{S}$, the space of lifts of $\mathbb{F}_p[X]$ to an $\mathbb{E}_\infty$ -coalgebra in the category of p -complete, connective spectra is contractible, with unique point given by $\mathbb{S}[X]_p^\wedge$. Moreover, for any $A \in \text{cCAlg}(\text{Sp}_p^{\text{cn}, \wedge})$ the base change along the map $\mathbb{S} \rightarrow \mathbb{F}_p$ induces an equivalence

$$\text{Map}_{\text{cCAlg}(\text{Sp}_p^{\text{cn}, \wedge})}(A, \mathbb{S}[X]_p^\wedge) \xrightarrow{\sim} \text{Map}_{\text{cCAlg}(\text{Mod}_{\mathbb{F}_p}^{\text{cn}})}(A \otimes \mathbb{F}_p, \mathbb{F}_p[X]).$$

¹For example, as a module the homology $\mathbb{F}_2[\mathbb{R}P^2]$ admits two different lifts to the 2-completed sphere, given by $\mathbb{S}[\mathbb{R}P^2]_2^\wedge$ and the split complex $\mathbb{S}_2^\wedge \oplus \Sigma^2 \mathbb{S}_2^\wedge$.

We prove Theorem A by inductively lifting first from $\mathbb{F}_p$ to $\mathbb{Z}_p$ along the p -completion tower and then from $\mathbb{Z}_p$ to $\mathbb{S}_p^\wedge$ along the Postnikov tower. A peculiarity of categories of coalgebras is that they admit somewhat mysterious right adjoints, and one of them plays a crucial role in our story. Suppose we have a map of connective $\mathbb{E}_\infty$ -rings $f: R' \rightarrow R$. Since the base change functor f^* is symmetric monoidal and preserves colimits, the adjoint functor theorem provides us with an adjunction:

$$f^*: \text{cCAlg}(\text{Mod}_{R'}^{\text{cn}}) \rightleftarrows \text{cCAlg}(\text{Mod}_R^{\text{cn}}) : f_*$$

We call f_* the *Weil restriction* along the map $f: R' \rightarrow R$.² Now given a connective R -module I , write $s_I: R \rightarrow R \oplus I$ for the 0-section of the split square zero extension with fiber I . We define a coalgebra

$$\mathfrak{D}_A(I) := (s_I)_*(s_I)^*A \in \text{cCAlg}(\text{Mod}_R^{\text{cn}})$$

as the Weil restriction of the trivial lift of A to a coalgebra over $R \oplus I$. This coalgebra comes equipped with a “universal derivation” $\pi_A: \mathfrak{D}_A(I) \rightarrow A$, as well as a section $\epsilon_A: A \rightarrow \mathfrak{D}_A(I)$ induced by the map $R \oplus I \rightarrow R$. This allows to define a class of coalgebras whose associated deformation problems are uniquely solvable.

Definition 1.1. A coalgebra $A \in \text{cCAlg}(\text{Mod}_R^{\text{cn}})$ is called *formally étale* if the universal derivation $\pi_A: \mathfrak{D}_A(I) \rightarrow A$ is an equivalence for any perfect connective R -module I . We denote by $\text{cCAlg}(\text{Mod}_R^{\text{cn}})^{\text{fét}}$ the full subcategory spanned by the formally étale coalgebras.

With this in hand, the proof of Theorem A breaks up into two parts. The first is to show that formally étale coalgebras lift uniquely and functorially against square zero extensions. More precisely, let k be a perfect $\mathbb{F}_p$ -algebra and denote by $\mathbb{W}(k)$ the spherical Witt vectors of k as in [Lur18b]. For an $\mathbb{E}_\infty$ -ring R denote by $\text{Mod}_R^\wedge$ the category of p -complete R -modules equipped with the symmetric monoidal structure given by the p -completed tensor product. We show that Weil restriction of formally étale coalgebras along $\mathbb{W}(k) \rightarrow k$ behaves like a coalgebraic version of the spherical Witt vector construction.

Theorem B. Weil restriction along the natural map $\mathbb{W}(k) \rightarrow k$ defines a fully faithful functor

$$\mathcal{R}_{\mathbb{W}(k)/k}: \text{cCAlg}(\text{Mod}_k^{\text{cn}})^{\text{fét}} \rightarrow \text{cCAlg}(\text{Mod}_{\mathbb{W}(k)}^{\text{cn}, \wedge})^{\text{fét}},$$

such that for any $A \in \text{cCAlg}(\text{Mod}_k^{\text{cn}})^{\text{fét}}$ the Weil restriction $\mathcal{R}_{\mathbb{W}(k)/k}(A)$ is, up to contractible choice, the unique coalgebra in $\text{Mod}_{\mathbb{W}(k)}^{\wedge, \text{cn}}$ with an equivalence $\mathcal{R}_{\mathbb{W}(k)/k}(A) \otimes k \simeq A$.

In fact, a variant of this holds very generally for Weil restriction of formally étale coalgebras along any pro-nilpotent thickening of connective $\mathbb{E}_\infty$ -rings $R' \rightarrow R$, see Proposition 4.19 for a precise statement.

The next step is to verify that the k -chains of any space $X \in \mathcal{S}$ are formally étale. We show that this claim can be reduced to the case where $X = \Omega^\infty \Sigma^n \mathbb{F}_p$ is an Eilenberg–MacLane space. In this situation, work of Bachmann–Burklund [BB24] shows that for any field k of characteristic p and any $n \geq 1$ we have a pullback square in $\text{cCAlg}(\text{Mod}_k^{\text{cn}})$ of the form:

$$\begin{array}{ccc} k[\Omega^\infty \Sigma^n \mathbb{F}_p] & \longrightarrow & C_k(\Sigma^n k) \\ \downarrow & & \downarrow \text{id} - F \\ k & \longrightarrow & C_k(\Sigma^n k). \end{array}$$

Here C_k denote the cofree coalgebra and $\text{id} - F$ is the Artin–Schreier map parameterizing the power operation $\text{id} - P^0$ acting on $\pi_{-n} A^\vee$ of any k -coalgebra A . This is the dual picture to the pushout square on cohomology of Mandell in [Man01], which was utilized by Lurie in [Lur11] to prove that the $\mathbb{F}_p$ -cohomology of an arbitrary space is formally étale. The pullback diagram tells

²Note that, since restriction of scalars is in general only *lax* monoidal, the right adjoint is not computed underlying.

us that the homology of an Eilenberg–MacLane space is cofree up to one “co-relation”. Since Weil restriction preserves cofree coalgebras, we show that the co-relation $F = \text{id}$ forces the coalgebra $k[\Omega^\infty \Sigma^n \mathbb{F}_p]$ to be formally étale, which allows us to prove our third theorem.

Theorem C. For any field k of characteristic p and any space $X \in \mathcal{S}$ the coalgebra $k[X] \in \text{cCAlg}(\text{Mod}_k^{\text{cn}})$ is formally étale.

Having seen this, our Theorem A is now immediate by combining Theorem B with Theorem C and in fact still holds if we replace $\mathbb{F}_p$ by any perfect field k of characteristic p and the sphere $\mathbb{S}_p^\wedge$ by the spherical Witt vectors $\mathbb{W}(k)$.

Recall that a space $X \in \mathcal{S}$ is called *p-complete* if it is local with respect to the functor $\mathbb{F}_p[-] : \mathcal{S} \rightarrow \text{Mod}_{\mathbb{F}_p}$ and denote by $\mathcal{S}_p \subseteq \mathcal{S}$ the full subcategory spanned by the *p-complete* spaces. Moreover, call a space *nilpotent* if for each $x \in \pi_0 X$ the fundamental group $\pi_1(X, x)$ is a nilpotent group which acts nilpotently on the higher homotopy groups. Write $\mathcal{S}_p^{\text{nilp}}$ for the category of *p-complete nilpotent* spaces. Combining Theorem A with [BB24, Theorem 1.2] we also deduce the following spherical, coalgebraic variant of the main theorem of [Man01]:

Corollary 1.2. Let k be a perfect, separably closed field of characteristic p with spherical Witt vectors $\mathbb{W}(k)$. The *p-complete* $\mathbb{W}(k)$ -homology functor

$$\mathcal{S}_p^{\text{nilp}} \rightarrow \text{cCAlg}(\text{Mod}_{\mathbb{W}(k)}^\wedge) \quad X \mapsto \mathbb{W}(k)[X]_p^\wedge$$

is fully faithful. In particular, for any nilpotent space X we have a natural equivalence

$$X_p^\wedge \simeq \text{Map}_{\text{cCAlg}_{\mathbb{W}(k)}^\wedge}(\mathbb{W}(k), \mathbb{W}(k)[X]_p^\wedge).$$

It would be pleasant if we could replace $\mathbb{W}(k)$ by $\mathbb{S}_p^\wedge$ in Corollary 1.2 as well. However, on the nose this not possible as we need to keep track of descent data. In upcoming work, we plan to identify this descent datum with the *coalgebra Frobenius*, a dual notion to the Tate Frobenius of [NS17] and construct an embedding of $\mathcal{S}_p^{\text{nilp}}$ into coalgebras $\text{cCAlg}(\text{Sp}_p^\wedge)$ equipped with a trivialization of the Frobenius action. This would dualize work of Yuan carried out in [Yua19] and allow for dropping the finiteness assumptions necessary therein.

Outline

We proceed along the following structure: In Section 1 we recall the definition and basic properties of $\mathbb{E}_\infty$ -coalgebras and discuss some relevant coalgebraic right adjoints. In Section 2 we review the setup of deformation theory developed by Lurie in [Lur12] and [Lur17]. We recall the notions of square zero extensions of $\mathbb{E}_\infty$ -rings and discuss cohesive and nilcomplete functors. We then define the tangent complex of a cohesive functor and give proofs of some important facts about its behavior. In Section 3 we prove that the moduli of coalgebras are cohesive and thus we can analyze them using the machinery discussed in the previous section. We then introduce formally étale coalgebras and prove that they can be lifted uniquely and functorially against square zero extensions. In Section 4 we show that the $\mathbb{F}_p$ -homology homology of any space is formally étale.

Conventions

Throughout the text, we use the following conventions:

1. By category we always mean $(\infty, 1)$ -category and refer to $(1, 1)$ -categories as 1-categories.
2. We choose three nested Grothendieck universes $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}$, and refer to categories built from them as small categories, categories and large categories, respectively. We denote by Cat_∞ the large category of categories and disregard size issues from here on out.
3. We denote the category of spaces by $\mathcal{S}$ and the category of spectra by Sp .

4. If A, B are objects in some category $\mathcal{C}$, we use the words map and morphism $A \rightarrow B$ interchangeably to mean a point in the mapping space $\text{Map}_{\mathcal{C}}(A, B) \in \mathcal{S}$. If moreover $\mathcal{C}$ is a stable category, we regard it as enriched over the category of spectra and write $\text{map}_{\mathcal{C}}(A, B) \in \text{Sp}$ for the mapping spectrum.
5. By (co)algebra we always mean $\mathbb{E}_{\infty}$ -(co)algebra. If we want to refer to a 1-categorical version of some gadget we call them *discrete*.
6. For a category $\mathcal{C}$ and some $n \geq 0$, we write $\mathcal{C}^{\Delta^n} := \text{Map}_{\text{Cat}_{\infty}}(\Delta^n, \mathcal{C})$ for the space of n -simplices in $\mathcal{C}$.

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2 Generalities on coalgebras

In this section, we recall the definition $\mathbb{E}_{\infty}$ -coalgebra in a symmetric monoidal category $\mathcal{C}$ and collect some general facts about categories of coalgebras. Crucially, if $\mathcal{C}$ is a presentably symmetric monoidal category, then the category of coalgebras $\text{cCAlg}(\mathcal{C})$ is presentable. Since the forgetful functor $\text{cCAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ moreover commutes with colimits, for any colimit preserving, symmetric monoidal functor $\mathcal{C} \rightarrow \mathcal{D}$, the induced functor $\text{cCAlg}(\mathcal{C}) \rightarrow \text{cCAlg}(\mathcal{D})$ admits an “unusual” right adjoint. We discuss several examples of these adjoints. Most crucially for us is the case where the left adjoint is given by base change along a map of (connective) $\mathbb{E}_{\infty}$ -rings, in which case we refer to the right adjoint as *Weil-restriction*.

Coalgebras in an arbitrary symmetric monoidal category are defined in terms of algebras in the opposite category. In light of this, let us begin by gathering some basic facts about algebras.

Definition 2.1. For any symmetric monoidal category $\mathcal{C}$, we denote by $\text{CAlg}(\mathcal{C})$ the category of $\mathbb{E}_{\infty}$ -algebras in $\mathcal{C}$. For $\mathcal{C} = \text{Sp}$ we write $\text{CAlg}(\text{Sp}) = \text{CAlg}$ and refer to objects of CAlg simply as $\mathbb{E}_{\infty}$ -rings.

Proposition 2.2. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category, then the following statements hold:

1. The forgetful functor $U_{\mathcal{C}}: \text{CAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ is conservative and commutes with limits.
2. The coproduct of two algebras $R, S \in \text{CAlg}(\mathcal{C})$ is given by the tensor product $R \otimes S$.
3. If $\mathcal{C}$ is presentable and $- \otimes -$ commutes with colimits in both variables separately, then $\text{CAlg}(\mathcal{C})$ is presentable as well.

Proof. The first claim is a combination of [Lur17, Lemma 3.2.2.6] and [Lur17, Corollary 3.2.2.5]. The second is shown in [Lur17, Corollary 3.2.4.7] and the third is [Lur17, Corollary 3.2.3.5]. $\square$

Definition 2.3. Let $\mathcal{C}$ be a symmetric monoidal category and equip its opposite category $\mathcal{C}^{\text{op}}$ with the inherited monoidal structure. The category of $\mathbb{E}_{\infty}$ -coalgebras in $\mathcal{C}$ is defined as

$$\text{cCAlg}(\mathcal{C}) := \text{CAlg}(\mathcal{C}^{\text{op}})^{\text{op}}.$$

Lemma 2.4. Let $\mathcal{C}$ be any symmetric monoidal category, then the following hold:

1. The forgetful functor $U_{\mathcal{C}}: \text{cCAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ is conservative and commutes with colimits.
2. The product of two coalgebras $A, B \in \text{cCAlg}(\mathcal{C})$ is given by $A \otimes B$.
3. If $\mathcal{C}$ is presentable and $- \otimes -$ commutes with colimits in each variable separately, then $\text{cCAlg}(\mathcal{C})$ is presentable.

Proof. The first two claims are immediate from Proposition 2.2 and the third is the content of [Lur18a, Corollary 3.1.4]. $\square$

The third claim in Lemma 2.4 is in some sense an analogue of the classical theorem by Sweedler that every coalgebra in the 1-category of vector spaces over a field is a filtered colimit of its finite dimensional sub-coalgebras, see [Swe69]. However, unlike in the classical situation, we cannot deduce that $\text{cCAlg}(\mathcal{C})$ is compactly generated if $\mathcal{C}$ was compactly generated. For example, as discussed in [Lur18b, Warning 1.2.15], the terminal $\mathbb{Q}$ -coalgebra $\mathbb{Q}$ is *not* a compact object of $\text{cCAlg}(\text{Mod}_{\mathbb{Q}})$. This is one of the main defects the category of coalgebras has over the category of algebras.

Definition 2.5. Let $\mathcal{C}$ be a presentable, symmetric monoidal category such that the tensor product commutes with colimits in each variable separately. Since by Lemma 2.4 the forgetful functor $U_{\mathcal{C}}: \text{cCAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ commutes with colimits and $\text{cCAlg}(\mathcal{C})$ is presentable, [Lur08, Theorem 5.5.2.9] implies that it admits a right adjoint

$$C_{\mathcal{C}}: \mathcal{C} \rightarrow \text{cCAlg}(\mathcal{C}).$$

For any $E \in \mathcal{C}$ we call $C_{\mathcal{C}}(E)$ the *cofree coalgebra* on E .

We will also make repeated use of the following lemma, which says that forming the category of coalgebras commutes with limits of categories.

Lemma 2.6. Let $\text{Cat}_{\infty}^{\otimes}$ denote the (large) category of symmetric monoidal categories and symmetric monoidal functors between them. The functors

$$\text{Cat}_{\infty}^{\otimes} \rightarrow \text{Cat}_{\infty} \quad \mathcal{C} \mapsto \text{CAlg}(\mathcal{C}),$$

$$\text{Cat}_{\infty}^{\otimes} \rightarrow \text{Cat}_{\infty} \quad \mathcal{C} \mapsto \text{cCAlg}(\mathcal{C})$$

commute with limits.

Proof. The first follows from the fact that the category $\text{CAlg}(\mathcal{C})$ is naturally identified with the category of symmetric monoidal functors $\text{Fun}^{\otimes}(\mathbb{E}_{\infty}^{\otimes}, \mathcal{C})$ where $\mathbb{E}_{\infty}^{\otimes}$ denotes the monoidal envelope of the $\mathbb{E}_{\infty}$ -operad. For the second functor, we simply observe that it is given by the composition

$$\text{Cat}_{\infty}^{\otimes} \xrightarrow{(-)^{\text{op}}} \text{Cat}_{\infty}^{\otimes} \xrightarrow{\text{CAlg}(-)} \text{Cat}_{\infty} \xrightarrow{(-)^{\text{op}}} \text{Cat}_{\infty},$$

which immediately implies the claim, since $(-)^{\text{op}}$ is an equivalence of categories. $\square$

Definition 2.7. Let Pr^{L} denote the (large) category of presentable categories with colimit preserving functors between them. We equip the category Pr^{L} with the Lurie tensor product, see [Lur17, Proposition 4.8.1.15], and call $\text{CAlg}(\text{Pr}^{\text{L}})$ the category of *presentably symmetric monoidal* categories.

Lemma 2.8. The natural functors $\text{CAlg}(\text{Pr}^{\text{L}}) \rightarrow \text{Pr}^{\text{L}} \rightarrow \text{Cat}_{\infty}$ commute with limits.

Proof. The first is an instance of Proposition 2.2 and the second is [Lur08, Proposition 5.5.3.13]. $\square$

Since colimits of coalgebras are computed underlying and the categories of coalgebras in presentably monoidal categories are again presentable by Lemma 2.4, colimit preserving symmetric monoidal functors often admit interesting³ right adjoint functors on the level of coalgebras. Let us discuss some examples of this phenomenon.

Example 2.9 (Connective covers). Let $\mathcal{C} \in \mathbf{CAlg}(\mathbf{Pr}^L)$ be stable and equipped with a t -structure, such that $\mathcal{C}_{\geq 0}$ is again presentable and closed under colimits and tensor products. Then, the inclusion functor $i: \mathcal{C}_{\geq 0} \rightarrow \mathcal{C}$ defines a map in $\mathbf{CAlg}(\mathbf{Pr}^L)$. Thus, it induces a functor $i: \mathbf{cCAlg}(\mathcal{C}_{\geq 0}) \rightarrow \mathbf{cCAlg}(\mathcal{C})$ that in turn by [Lur08, Theorem 5.5.2.9] admits a right adjoint, which we denote

$$\tau_{\geq 0}^{\mathcal{C}}: \mathbf{cCAlg}(\mathcal{C}) \rightarrow \mathbf{cCAlg}(\mathcal{C}_{\geq 0}).$$

Since connective coalgebras form a full subcategory, it is true that $\tau_{\geq 0}^{\mathcal{C}}(A) \simeq A$ whenever $U_{\mathcal{C}}(A) \in \mathcal{C}_{\geq 0}$. In general, $U_{\mathcal{C}}(\tau_{\geq 0}^{\mathcal{C}} A)$ need not agree with the underlying connective cover $\tau_{\geq 0} U_{\mathcal{C}}(A)$.

Note that, since the inclusion of the connective coalgebras is computed underlying, the coalgebraic connective cover commutes with cofree coalgebras. More precisely, we have that $\tau_{\geq 0}^{\mathcal{C}} C_{\mathcal{C}}(E) \simeq C_{\mathcal{C}_{\geq 0}}(\tau_{\geq 0} E)$ for any $E \in \mathcal{C}$.

Example 2.10 (Weil restriction). Fix some functor $\mathcal{C}_-: \mathbf{CAlg}^{\text{cn}} \rightarrow \mathbf{CAlg}(\mathbf{Pr}^L)$ and write $f^*: \mathcal{C}_R \rightarrow \mathcal{C}_S$ for the functor induced by a map of connective $\mathbb{E}_{\infty}$ -rings $f: R \rightarrow S$. Since f^* is a map in $\mathbf{CAlg}(\mathbf{Pr}^L)$, it admits a lax symmetric monoidal right adjoint $f_!: \mathcal{C}_S \rightarrow \mathcal{C}_R$ ⁴. We think of $f^* \dashv f_!$ respectively as *base change* and *restriction* along f . Moreover, the induced functor on coalgebras $f^*: \mathbf{cCAlg}(\mathcal{C}_R) \rightarrow \mathbf{cCAlg}(\mathcal{C}_S)$ also admits a right adjoint

$$f_*: \mathbf{cCAlg}(\mathcal{C}_S) \rightarrow \mathbf{cCAlg}(\mathcal{C}_R),$$

which we call *Weil restriction* along f .

Note again that, since f^* is computed underlying, f_* preserves cofree coalgebras, meaning for every $E \in \mathcal{C}_S$ we have a natural equivalence $f_* C_S(E) \simeq C_R(f_! E)$. In particular, for the terminal coalgebra $\mathbb{1}_{\mathcal{C}_S}$ we have that $f_* \mathbb{1}_{\mathcal{C}_S} \simeq \mathbb{1}_{\mathcal{C}_R}$.

Example 2.11 (Duality). For any $\mathcal{C} \in \mathbf{CAlg}(\mathbf{Pr}^L)$ write

$$\text{map}_{\mathcal{C}}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$$

for the internal mapping object functor. This functor inherits a natural lax monoidal structure with respect to the pointwise monoidal structure on $\mathcal{C}^{\text{op}} \times \mathcal{C}$ and thus induces a functor

$$\text{map}_{\mathcal{C}}(-, -): \mathbf{CAlg}(\mathcal{C}^{\text{op}} \times \mathcal{C}) \simeq \mathbf{cCAlg}(\mathcal{C})^{\text{op}} \times \mathbf{CAlg}(\mathcal{C}) \rightarrow \mathbf{CAlg}(\mathcal{C}).$$

In particular, evaluating at the unit $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ and applying the involution $(-)^{\text{op}}$, we get a functor

$$(-)^{\vee} := \text{map}_{\mathcal{C}}(-, \mathbb{1}_{\mathcal{C}}): \mathbf{cCAlg}(\mathcal{C}) \rightarrow \mathbf{CAlg}(\mathcal{C})^{\text{op}}.$$

Since colimits of coalgebras and limits of algebras are computed underlying, the functor $(-)^{\vee}$ preserves colimits. Thus, by the adjoint functor theorem [Lur08, Theorem 5.5.2.9], we get an adjunction:

$$(-)^{\vee}: \mathbf{cCAlg}(\mathcal{C}) \rightleftarrows \mathbf{CAlg}(\mathcal{C})^{\text{op}}: (-)^{\circ}.$$

Given $A \in \mathbf{cCAlg}(\mathcal{C})$ and $R \in \mathbf{CAlg}(\mathcal{C})$, we call $A^{\vee}$ the *dual algebra* of A and R° the *pre-dual coalgebra* of R .

If the underlying object of $R \in \mathbf{CAlg}(\mathcal{C})$ is dualizable, then the dual $R^{\vee}$ also inherits a natural coalgebra structure. In fact, this construction agrees with R° as we will see in Corollary 2.14.

³In the sense that they are not computed underlying.

⁴This notation is somewhat non-standard and unfortunate, however the functor $f_!$ will not feature prominently in what follows. Hence, we opt to use the more natural f_* for the coalgebraic right adjoint instead.

Proposition 2.12. Let $\mathcal{C}$ be a symmetric monoidal category and denote by $\mathcal{C}^\diamond$ the full subcategory spanned by the dualizable objects. The functor

$$\mathrm{cAlg}(\mathcal{C}^\diamond)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{C}^\diamond) \quad A \mapsto A^\vee$$

is an equivalence of categories with inverse taking $R \in \mathrm{CAlg}(\mathcal{C}^\diamond)$ to the dual $R^\vee$ with the induced coalgebra structure.

Proof. This is immediate from [Lur18a, Proposition 3.2.4]. $\square$

For a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ and an object $A \in \mathcal{C}$, say that F is *fully faithful at A* if the natural transformation $\mathrm{Map}_{\mathcal{C}}(A, -) \rightarrow \mathrm{Map}_{\mathcal{D}}(F(A), F(-))$ is an equivalence.

Lemma 2.13. Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a lax symmetric monoidal functor. Suppose we have $R \in \mathrm{CAlg}(\mathcal{C})$ such that F is fully faithful at each tensor power $R^{\otimes n} \in \mathcal{C}$ for $n \geq 0$. Then the natural map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(R, S) \xrightarrow{F} \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D})}(F(R), F(S))$$

is an equivalence.

Proof. Since F is lax symmetric monoidal, it induces a map of ∞ -operads

$$\begin{array}{ccc} \mathcal{C}^\otimes & \xrightarrow{F^\otimes} & \mathcal{D}^\otimes \\ & \searrow p \quad \swarrow q & \\ & \mathrm{Fin}_* & \end{array}$$

which takes any $(A_1, \dots, A_n) \in \mathcal{C}_{\langle n \rangle}^\otimes$ to the point $(F(A_1), \dots, F(A_n)) \in \mathcal{D}_{\langle n \rangle}^\otimes$. Write $\Gamma(p)$ and $\Gamma(q)$ for the category of sections of p and q , respectively. With this notation, a commutative algebra structure on some $R \in \mathcal{C}$ is precisely given by a section $s_R \in \Gamma(p)$ which takes $\langle n \rangle$ to $(R, \dots, R) \in \mathcal{C}_{\langle n \rangle}$ and maps inert morphisms to inert morphisms. Let $\varphi: \langle n \rangle \rightarrow \langle m \rangle$ be a morphism in Fin_* and denote by $p_i: \langle m \rangle \rightarrow \langle 1 \rangle$ the unique inert map which sends $i \mapsto 1$. For each i we have a factorization

$$\langle n \rangle \xrightarrow{\psi_i} \langle k_i \rangle \xrightarrow{\pi_i} \langle 1 \rangle$$

of $p_i \circ \varphi$ into an inert map ψ_i and an active map π_i . Then for each $B = (B_1, \dots, B_m) \in \mathcal{C}^\otimes$ we get equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}^\otimes}^\varphi(s_R(\langle n \rangle), B) &\simeq \prod_{i=1, \dots, m} \mathrm{Map}_{\mathcal{C}^\otimes}^{p_i \circ \varphi}((R, \dots, R), B_i) \\ &\simeq \prod_{i=1, \dots, m} \mathrm{Map}_{\mathcal{C}}(R^{\otimes k_i}, B_i) \\ &\simeq \prod_{i=1, \dots, m} \mathrm{Map}_{\mathcal{D}}(F(R)^{\otimes k_i}, F(B_i)) \\ &\simeq \prod_{i=1, \dots, m} \mathrm{Map}_{\mathcal{D}^\otimes}^{p_i \circ \varphi}((F^\otimes \circ s_R)(\langle n \rangle), F(B_i)) \\ &\simeq \mathrm{Map}_{\mathcal{D}^\otimes}^\varphi((F^\otimes \circ s_R)(\langle n \rangle), f(B)), \end{aligned}$$

hence $F^\otimes$ is faithful at each value of the section $s_R: \mathrm{Fin}_* \rightarrow \mathcal{C}^\otimes$. Consequently, the induced functor $F_*^\otimes: \Gamma(p) \rightarrow \Gamma(q)$ is fully faithful at $s_R \in \Gamma(p)$ and since $\mathrm{CAlg}(\mathcal{C})$ and $\mathrm{CAlg}(\mathcal{D})$ are full subcategories of $\Gamma(p)$ and $\Gamma(q)$ respectively, the claim follows. $\square$

Corollary 2.14. Let $\mathcal{C}$ be a symmetric monoidal category and $A, B \in \mathrm{cAlg}(\mathcal{C})$ with A dualizable. The natural map

$$\mathrm{Map}_{\mathrm{cAlg}(\mathcal{C})}(B, A) \rightarrow \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A^\vee, B^\vee)$$

is an equivalence. In particular, we have that $(A^\vee)^\circ \simeq A$.

Proof. Since the functor $(-)^{\vee}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ is fully faithful at any dualizable object and dualizable objects are closed under tensor products, the claim follows by applying Lemma 2.13 $\square$

Lemma 2.15. Let $\mathcal{C}$ be a category which admits finite products equipped with the cartesian monoidal structure. Then the forgetful functor $U_{\mathcal{C}}: \text{cCAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ is an equivalence.

Proof. By [Lur17, Corollary 2.4.4.10] the map $\text{CAlg}(\mathcal{C}^{\text{op}}) \rightarrow \mathcal{C}^{\text{op}}$ is an equivalence, hence the claim follows by applying $(-)^{\text{op}}$. $\square$

Example 2.16. By definition, any $\mathcal{C} \in \text{CAlg}(\text{Pr}^{\text{L}})$ comes equipped with a symmetric monoidal, colimit preserving functor

$$\mathbb{1}_{\mathcal{C}}[-]: \mathcal{S} \rightarrow \mathcal{C},$$

which takes a space X to the colimit over the constant diagram $\mathbb{1}_{\mathcal{C}}[X] = \text{colim}_X \mathbb{1}_{\mathcal{C}}$. Thus, by Lemma 2.15, we obtain an induced functor

$$\mathbb{1}_{\mathcal{C}}[-]: \mathcal{S} \xrightarrow{\sim} \text{cCAlg}(\mathcal{S}) \rightarrow \text{cCAlg}(\mathcal{C}).$$

Applying this to the presentably symmetric monoidal category Mod_R where R is some $\mathbb{E}_{\infty}$ -ring, we learn that the R -homology $R[X]$ of a space X carries a natural coalgebra structure. Moreover, the R -cohomology of X

$$R[X]^{\vee} \simeq \lim_X R = R^X$$

inherits a natural R -algebra structure. If X is a compact space, then $R[X]$ is a compact R -module and hence dualizable, so by Corollary 2.14 we have a natural equivalence of coalgebras $(R^X)^{\circ} \simeq R[X]$.

3 Review of deformation theory

In this section, we review the theory of square zero extensions of $\mathbb{E}_{\infty}$ -rings and the associated deformation theory of functors $X: \text{CAlg}^{\text{cn}} \rightarrow \mathcal{S}$, as developed by Lurie in [Lur17] and [Lur12].

Recall that, a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called:

1. *reduced* if $\mathcal{C}$ admits a terminal object $*$ in $\mathcal{C}$ and $F(*) \in \mathcal{D}$ is terminal.
2. *excisive* if $\mathcal{C}$ admits pushouts and F takes pushouts to pullbacks.

For a category $\mathcal{C}$ which admits finite limits, write $\text{Sp}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathcal{S}_{*}^{\text{fin}}, \mathcal{C})$ spanned by the reduced and excisive functors, where $\mathcal{S}_{*}^{\text{fin}}$ is the category of finite pointed spaces. Moreover, for any $n \geq 0$ denote by $\Omega^{\infty-n}: \text{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$ the functor given by evaluation at $S^n \in \mathcal{S}$.

Now let R be a connective $\mathbb{E}_{\infty}$ -ring. By [Lur17, Theorem 7.3.4.13] the assignment taking a map of $\mathbb{E}_{\infty}$ -rings $R' \rightarrow R$ to its fiber induces an equivalence of categories

$$\text{Sp}(\text{CAlg}_{/R}) \xrightarrow{\sim} \text{Mod}_R$$

such that for each $n \geq 0$ the composite functor

$$\text{Mod}_R \xrightarrow{\sim} \text{Sp}(\text{CAlg}_{/R}) \xrightarrow{\Omega^{\infty-n}} \text{CAlg}_{/R}$$

takes a module I to an augmented R -algebra, whose underlying module is given by the direct sum $R \oplus \Sigma^n I$. We implicitly equip $R \oplus \Sigma^n I$ with this algebra structure from here on out.

Definition 3.1. We say that a map of connective $\mathbb{E}_{\infty}$ -rings $R^{\eta} \rightarrow R$ is a *square zero extension* of R with fiber $I \in \text{Mod}_R^{\text{cn}}$ if it fits into a pullback diagram of $\mathbb{E}_{\infty}$ -rings

$$\begin{array}{ccc} R^{\eta} & \longrightarrow & R \\ \downarrow & & \downarrow (\text{id}, 0) \\ R & \xrightarrow{(\text{id}, \eta)} & R \oplus \Sigma I. \end{array}$$

Moreover, we say that $R^\eta \rightarrow R$ is *split*, if the map (id, η) is homotopic to $(\text{id}, 0)$, i.e. $R^\eta \simeq R \oplus I$ and call the mapping space $\text{Map}_{\text{CAlg}/R}(R, R \oplus I)$ the space of *derivations* $\eta: R \rightarrow I$. Finally, we call a map of connective $\mathbb{E}_\infty$ -rings a *nilpotent thickening* if it can be written as a finite composite of square zero extensions whose fibers are perfect modules.

Remark 3.2. If $p: R^\eta \rightarrow R$ is any square zero extension classified by a derivation $\eta: R \rightarrow \Sigma I$, then by taking fibers we get a commutative diagram with exact columns

$$\begin{array}{ccc} I & \xrightarrow{\sim} & I \\ \downarrow & & \downarrow 0 \\ R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow (\text{id}, 0) \\ R & \xrightarrow{(\text{id}, \eta)} & R \oplus \Sigma I. \end{array}$$

In particular, the fiber of f inherits a natural R -module structure, such that the multiplication map factors as

$$I \otimes_{R^\eta} I \rightarrow I \otimes_R I \rightarrow I$$

where the second map is induced from the null map $0: I \rightarrow R$ by applying $- \otimes_R I$. Since any map of the form $f_1 \otimes \cdots \otimes f_{n-1} \otimes 0$ admits a Σ_n -equivariant nullhomotopy, we deduce that all coherent multiplication maps

$$(I^{\otimes n})_{h\Sigma_n} \rightarrow I$$

are nullhomotopic. In this sense, square zero extensions of $\mathbb{E}_\infty$ -rings are “coherently” square zero and admit no nontrivial power operations on the fiber.

To make this definition work for us we require a way to check in practice whether a given map of $\mathbb{E}_\infty$ -rings is a square zero extension. This is provided by the following proposition, which in particular implies that for discrete rings our notion agrees with the classical definition of a square zero extension.

Proposition 3.3. Let $R' \rightarrow R$ be a map of connective $\mathbb{E}_\infty$ -rings with fiber I such that $I \in \text{Sp}_{\geq n} \cap \text{Sp}_{\leq 2n}$ and the multiplication map $I \otimes_{R'} I \rightarrow I$ is nullhomotopic. Then $R' \rightarrow R$ is a square zero extension.

Proof. This is immediate from [Lur17, Theorem 7.4.1.23]. $\square$

Example 3.4. Let $f: R' \rightarrow R$ be a surjective map of ordinary commutative rings with kernel $I \subseteq R'$. Then $f: R' \rightarrow R$ is a square zero extension if and only if $I^2 = 0$. In particular, for every $n \geq 1$ the map $\mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^{n-1}$ is square zero with fiber $\mathbb{F}_p$.

Example 3.5. For any connective $\mathbb{E}_\infty$ -ring R and any $n \geq 1$, the map $\tau_{\leq n} R \rightarrow \tau_{\leq n-1} R$ is a square zero extension with fiber $\Sigma^n \pi_n R$.

Given a functor $X: \text{CAlg}^{\text{cn}} \rightarrow \mathcal{S}$ and a square zero extension $R^\eta \rightarrow R$ we want to study the fibers of the map $X(R^\eta) \rightarrow X(R)$, i.e. given $A \in X(R)$ we wish to understand the space of deformations of A to an object $\tilde{A} \in X(R^\eta)$. Notice that, since R and I are connective, the derivation

$$R \xrightarrow{(\text{id}, \eta)} R \oplus \Sigma I$$

is surjective on π_0 , indeed an isomorphism. This motivates the following definition for a class of functors which are “well behaved” with regard to square zero extensions.

Definition 3.6. Let $\mathcal{C}$ be a category that admits all limits. A functor $X: \text{CAlg}^{\text{cn}} \rightarrow \mathcal{C}$ is called:

1. *Cohesive* if for any pullback diagram of connective $\mathbb{E}_\infty$ -rings

$$\begin{array}{ccc} R' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

which induces surjections $\pi_0 R \rightarrow \pi_0 S$ and $\pi_0 S' \rightarrow \pi_0 S$, the diagram

$$\begin{array}{ccc} X(R') & \longrightarrow & X(S') \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(S) \end{array}$$

is a pullback.

2. *Nilcomplete* if for every connective $\mathbb{E}_\infty$ -ring R the natural map

$$X(R) \rightarrow \varprojlim_n X(\tau_{\leq n} R)$$

is an equivalence.

Example 3.7. For any connective $\mathbb{E}_\infty$ -ring R the functor

$$\mathrm{Spec}(R): \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{S}, \quad S \mapsto \mathrm{Map}_{\mathrm{CAlg}}(R, S)$$

commutes with all limits, and so in particular it is cohesive and nilcomplete.

Definition 3.8. Let $X: \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{C}$ be a cohesive functor. Then any $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and any point $A \in X(R)$ gives rise to a *moduli problem*, namely the functor

$$X_A^-: \mathrm{CAlg}_{/R}^{\mathrm{cn}} \rightarrow \mathcal{C}, \quad (R' \rightarrow R) \mapsto \mathrm{fib}_A(X(R') \rightarrow X(R)).$$

For any $(R' \rightarrow R) \in \mathrm{CAlg}_{/R}^{\mathrm{cn}}$, we call $X_A^{R'}$ the *space of deformations* of A along the map $R' \rightarrow R$.

The moduli problems associated to a cohesive functor give rise to spectrum objects, by the following observation:

Lemma 3.9. Let $\mathcal{C}$ be a category which admits finite limits, $X: \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{C}$ be a cohesive functor and $A \in X(R)$ be an R -valued point. The functor given by the composition

$$X_A^{R \oplus -}: \mathrm{Mod}_R^{\mathrm{cn}} \xrightarrow{\Omega^\infty|_{\mathrm{Mod}_R^{\mathrm{cn}}}} \mathrm{CAlg}_{/R}^{\mathrm{cn}} \xrightarrow{X_A^-} \mathcal{C}$$

is reduced and excisive.

Proof. Clearly, we have $X_A^R \simeq *$, so the functor is reduced. By [Lur17, Proposition 1.4.2.13], to show that the functor is excisive, it suffices to check that it takes suspensions to desuspensions. By definition, the functor $\Omega^\infty|_{\mathrm{Mod}_R^{\mathrm{cn}}}$ takes any pushout diagram of the form

$$\begin{array}{ccc} I & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma I \end{array}$$

to the pullback diagram

$$\begin{array}{ccc} R \oplus I & \longrightarrow & R \\ \downarrow & & \downarrow (\mathrm{id}, 0) \\ R & \xrightarrow{(\mathrm{id}, 0)} & R \oplus \Sigma I. \end{array}$$

Since X is cohesive and taking fibers commutes with limits, we learn that $\Omega X_A^{R \oplus \Sigma I} \simeq X_A^{R \oplus I}$ for any $I \in \mathrm{Mod}_R^{\mathrm{cn}}$ and so we conclude. $\square$

Construction 3.10. Let $\mathcal{C}$ be a category which admits finite limits, let $X: \mathbf{CAlg}^{\text{cn}} \rightarrow \mathcal{C}$ be cohesive and $A \in X(R)$. We know by Lemma 3.9 that the functor $X_A^{R\oplus-}$ is reduced and excisive. Thus, by [Lur17, Proposition 1.4.2.22] we obtain a unique factorization

$$\begin{array}{ccc} & & \text{Sp}(\mathcal{C}) \\ & \nearrow T_{X_A}^- & \downarrow \Omega^\infty \\ \text{Mod}_R^{\text{cn}} & \xrightarrow{X_A^{R\oplus-}} & \mathcal{C} \end{array}$$

such that, for any $I \in \text{Mod}_R^{\text{cn}}$ and $n \geq 0$ we have a natural equivalence $\Omega^{\infty-n} T_{X_A}^I \simeq X_A^{R\oplus \Sigma^n I}$. We refer to the functor $T_{X_A}^-$ as the *tangent spectrum* of X at the point $A \in X(R)$. When it is clear from context, we also leave the functor X implicit and simply write $T_A^- = T_{X_A}^-$.

The tangent spectrum conveniently packages the data of a moduli problem into a single spectrum object as follows:

Proposition 3.11. Let $X: \mathbf{CAlg}^{\text{cn}} \rightarrow \mathcal{S}$ be a cohesive functor and $R^\eta \rightarrow R$ a square zero extension classified by a derivation $R \xrightarrow{\eta} \Sigma I$. Then for each $A \in X(R)$ the space of deformations $X_A^{R^\eta}$ is either empty or a torsor under the grouplike $\mathbb{E}_\infty$ -monoid $\Omega^\infty T_{X_A}^I$. Moreover, η determines an obstruction class in $\pi_{-1} T_{X_A}^I$, which vanishes if and only if $X_A^{R^\eta}$ is non-empty.

Proof. Since X is cohesive, applying X_A^- to the pullback diagram defining R^η :

$$\begin{array}{ccc} R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow 0 \\ R & \xrightarrow{(0, \eta)} & R \oplus \Sigma I, \end{array}$$

we get a pullback of spaces

$$\begin{array}{ccc} X_A^{R^\eta} & \longrightarrow & * \\ \downarrow & & \downarrow A^0 \\ * & \xrightarrow{A^\eta} & X_A^{R\oplus \Sigma I}, \end{array}$$

exhibiting $X_A^{R^\eta}$ as the space of paths in $X_A^{R\oplus \Sigma I}$ between the points A^0 and A^η . Hence, it is non-empty if and only if the homotopy class determined by the map

$$* \xrightarrow{A^\eta} X_A^{R\oplus \Sigma I} \simeq \Omega^\infty T_{X_A}^{\Sigma I}$$

vanishes. Moreover, in this case $X_A^{R^\eta}$ is a torsor under the loop space based at A^0 , which is given by

$$\Omega X_A^{R\oplus \Sigma I} \simeq X_A^{R\oplus I} \simeq \Omega^\infty T_{X_A}^{\Sigma I},$$

as claimed. $\square$

Proposition 3.12. Let R be a connective $\mathbb{E}_\infty$ -ring, $X: \mathbf{CAlg}^{\text{cn}} \rightarrow \mathcal{S}$ be cohesive and $A \in X(R)$. Then $T_{X_A}^R$ inherits a natural R -module structure such that for any perfect connective R -module I we have a natural equivalence $T_{X_A}^I \simeq T_{X_A} \otimes_R I$.

Proof. We know from Lemma 3.9 that the functor $F = X_{X_A}^{R\oplus-}: \text{Mod}_R^{\text{cn}} \rightarrow \mathcal{S}$ is excisive. Hence, by [Lur12, Lemma 1.3.2] it extends uniquely to an excisive functor $F': \text{Mod}_R^{\text{acn}} \rightarrow \mathcal{S}$. Since $\text{Mod}_R^{\text{acn}}$ is stable, F' is an exact functor. Thus, the restriction $F'|_{\text{Mod}_R^{\text{perf}}}$ is also exact. Hence, since Sp is stable, [Lur17, Proposition 1.4.2.22] implies that we get an essentially unique lift to

an exact functor $\tilde{F}: \text{Mod}_R^{\text{perf}} \rightarrow \text{Sp}$. Finally, applying [Lur08, Proposition 5.5.1.9] we see that $\tilde{F}$ induces a colimit preserving functor $\text{Ind}(\text{Mod}_R^{\text{perf}}) \simeq \text{Mod}_R \rightarrow \text{Sp}$, which under the equivalence

$$\text{Fun}^L(\text{Mod}_R, \text{Sp}) \simeq \text{Mod}_R \quad G \mapsto G(R)$$

yields a R -module whose underlying spectrum is given by $T_{X_A}^R$. $\square$

Proposition 3.13. Let $X: \text{CAlg}^{\text{cn}} \rightarrow \mathcal{S}$ be cohesive and $R' \rightarrow R$ a map of connective $\mathbb{E}_\infty$ -rings. Moreover, let $A' \in X(R')$ and write $A \in X(R)$ for the image under the induced map $X(R') \rightarrow X(R)$. Then for every $I \in \text{Mod}_R^{\text{cn}}$ we have a natural map

$$T_{X_{A'}}^I \rightarrow T_{X_A}^I,$$

which is an equivalence if the map $\pi_0 R' \rightarrow \pi_0 R$ is surjective.

Proof. Applying X to the pullback of connective $\mathbb{E}_\infty$ -rings

$$\begin{array}{ccc} R' \oplus I & \longrightarrow & R \oplus I \\ \downarrow & & \downarrow \\ R' & \longrightarrow & R \end{array}$$

and taking the fibers over the points $A \in X(R)$ and $A' \in X(R')$ gives a commutative diagram

$$\begin{array}{ccc} X_{A'}^{R' \oplus I} & \longrightarrow & X_A^{R \oplus I} \\ \downarrow & & \downarrow \\ X(R' \oplus I) & \longrightarrow & X(R \oplus I) \\ \downarrow & & \downarrow \\ X(R') & \longrightarrow & X(R) \end{array} \tag{1}$$

The map $X_{A'}^{R' \oplus I} \rightarrow X_A^{R \oplus I}$ is natural in M and thus induces a map of spectra $T_{X_{A'}}^I \rightarrow T_{X_A}^I$ as claimed. Moreover, if $R' \rightarrow R$ is surjective on π_0 the same is true for the map $R' \oplus I \rightarrow R$ and hence the lower square in the diagram (1) is a pullback square, as X was cohesive. In particular, the map on the fibers $X_{A'}^{R' \oplus I} \rightarrow X_A^{R \oplus I}$ is an equivalence and thus the induced map $T_{X_{A'}}^I \rightarrow T_{X_A}^I$ is as well. $\square$

Notice that, if $R^\eta \rightarrow R$ is a square zero extension of connective $\mathbb{E}_\infty$ -rings the map $\pi_0 R^\eta \rightarrow \pi_0 R$ is necessarily surjective. Thus, if we are given $A \in X(R)$ and a lift $A^\eta \in X(R^\eta)$, we know that if we have $M \in \text{Mod}_{R^\eta}^{\text{cn}}$ such that the R^η -action factors through R , then $T_{X_{A^\eta}}^M$ agrees with $T_{X_A}^M$.

As an example we now summarize the well-known deformation theory of $\mathbb{E}_\infty$ -rings in our language. Later we will use this description to analyze the deformation theory of dualizable coalgebras.

Example 3.14. Let R be a connective $\mathbb{E}_\infty$ -ring. The composition

$$\text{Mod}_R \xrightarrow{\Omega^\infty} \text{CAlg}/_R \xrightarrow{\text{Map}_{\text{CAlg}/_R}(R, -)} \mathcal{S}$$

is accessible and commutes with limits. Thus, since Mod_R is presentable, the adjoint functor theorem implies that it is corepresented by an R -Module L_R called the *cotangent complex* of R . Moreover, the functor

$$X := \text{Spec}(R): \text{CAlg}^{\text{cn}} \rightarrow \mathcal{S}, \quad S \mapsto \text{Map}_{\text{CAlg}}(R, S)$$

is cohesive by Example 3.7 and for any $(\varphi: R \rightarrow S) \in \text{Spec}(R)(S)$ and $I \in \text{Mod}_S^{\text{cn}}$ we have that

$$\begin{aligned} \text{fib}_\varphi(\text{Map}_{\text{CAlg}}(R, S \oplus I) \rightarrow \text{Map}_{\text{CAlg}}(R, S)) &\simeq \text{Map}_{\text{CAlg}_{/S}}(R, S \oplus I) \\ &\simeq \text{Map}_{\text{CAlg}_{/R}}(R, R \oplus I) \\ &\simeq \text{Map}_{\text{Mod}_R}(L_R, I) \\ &\simeq \text{Map}_{\text{Mod}_S}(L_R \otimes_R S, I). \end{aligned}$$

Explicitly, this tells us that the space of lifts in the diagram

$$\begin{array}{ccc} & S \oplus I & \\ & \nearrow \text{dashed} & \downarrow \\ R & \xrightarrow{\varphi} & S \end{array}$$

is naturally identified with $\Omega^\infty T_{X_\varphi}^I \simeq \text{Map}_S(L_R \otimes_R S, I)$. In particular, if $L_R \otimes_R S \simeq 0$, then by Proposition 3.11 the map $\varphi: R \rightarrow S$ lifts uniquely along any square zero extension $S' \rightarrow S$.

Example 3.15. Given a category $\mathcal{C}$ write $\mathcal{C}^{\Delta^0}$ for the underlying space of $\mathcal{C}$. We will see later in Proposition 4.1 that the functor

$$X: \text{CAlg}^{\text{cn}} \rightarrow \mathcal{S}, \quad R \mapsto (\text{CAlg}_R^{\text{cn}})^{\Delta^0}$$

is cohesive. Moreover, it follows from [Lur17, Proposition 7.4.2.5] that for every $S \in X(R)$ there exists an S -module $L_{S/R}$ called the *relative cotangent complex*, together with, for every connective R -module I a natural equivalence

$$\text{Map}_S(L_{S/R}, \Sigma I \otimes_R S) \xrightarrow{\sim} \text{fib}_S(X(R \oplus I) \rightarrow X(R)) = \Omega^\infty T_{X_S}^I.$$

Thus, the problem of lifting S to a $R \oplus M$ -algebra is equivalent to finding a map of R -algebras fitting into the diagram

$$\begin{array}{ccc} & S \oplus (S \otimes_R \Sigma I) & \\ & \nearrow \text{dashed} & \downarrow \\ S & \xrightarrow{\text{id}} & S. \end{array}$$

In particular, if $L_{S/R} \simeq 0$, then by Proposition 3.11 the R -algebra S lifts uniquely along any square zero extension of $R' \rightarrow R$. In this case, S is called a *formally étale* R -algebra.

The main goal of the next section will be to develop statements analogous to the ones in the two examples above for coalgebras.

4 Deformation theory of coalgebras

We now apply the machinery reviewed in the previous section to study deformation theoretic questions about coalgebras in the category of spectra. We begin by deducing that the functor which assigns to a connective $\mathbb{E}_\infty$ -ring R its category of connective, p -complete modules is cohesive and nilcomplete in the sense of Definition 3.6. We then introduce *formally étale coalgebras* in an arbitrary cohesive moduli problem

$$\mathcal{C}_-: \text{CAlg}^{\text{cn}} \rightarrow \text{CAlg}(\text{Pr}^{\text{L}})$$

and show that the space of lifts of a formally étale coalgebra $A \in \text{cCAlg}(\mathcal{C}_R)$ along any nilpotent thickening $R' \rightarrow R$ is contractible. Moreover, we show that the assignment of A to its essentially unique lift $A' \in \text{cCAlg}(\mathcal{C}_{R'})$ refines to a fully faithful functor $\text{cCAlg}(\mathcal{C}_R)^{\text{fét}} \rightarrow \text{cCAlg}(\mathcal{C}_{R'})^{\text{fét}}$.

4.1 Connective and p -complete moduli

Our starting point is the following result of Lurie:

Proposition 4.1. For any connective $\mathbb{E}_\infty$ -ring R the functor

$$\mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \quad R \mapsto \mathrm{Mod}_R^{\mathrm{cn}}$$

is cohesive and nilcomplete.

Proof. This is precisely [Lur21, Proposition 19.2.1.3] with the additional observation that the relevant comparison functors on almost connective modules preserve and detect connective modules. $\square$

Now fix a prime p and let R be an $\mathbb{E}_\infty$ -ring. Recall that a module $M \in \mathrm{Mod}_R$ is called p -complete if the limit

$$\lim \left(\dots \xrightarrow{p} M \xrightarrow{p} M \right)$$

vanishes. We denote the full subcategory spanned by the p -complete modules by $\mathrm{Mod}_R^\wedge$. The inclusion $\mathrm{Mod}_R^\wedge \rightarrow \mathrm{Mod}_R$ admits a left adjoint which takes a module M to its p -completion, given by the limit

$$M^\wedge := \lim \left(\dots \rightarrow M/p^2 \rightarrow M/p \right).$$

In fact, M is p -complete if and only if the natural map $M \rightarrow \lim M/p^n$ is an equivalence and so p -completion defines a localization of Mod_R . Moreover, the functor $(-)^\wedge$ admits a further left adjoint, which takes a p -complete R -module M to its corresponding torsion module $\mathrm{map}_R(R^{p\text{-tor}}, M)$ where $R^{p\text{-tor}} = \mathrm{fib}(R \rightarrow R/p)$, and hence preserves both limits and colimits. The category $\mathrm{Mod}_R^\wedge$ admits a natural presentably symmetric monoidal structure given by the formula

$$M \otimes_{\mathrm{Mod}_R^\wedge} N = (M \otimes_R N)^\wedge,$$

such that the p -completion functor $\mathrm{Mod}_R \rightarrow \mathrm{Mod}_R^\wedge$ is symmetric monoidal.

Lemma 4.2. Let $R_- : \mathcal{J} \rightarrow \mathrm{CAlg}^{\mathrm{cn}}$ be a diagram of connective $\mathbb{E}_\infty$ -rings with limit $R = \lim_i R_i \in \mathrm{CAlg}^{\mathrm{cn}}$. Then the natural comparison functor

$$F : \mathrm{Mod}_R^{\mathrm{cn}} \rightarrow \varprojlim_{i \in \mathcal{J}} \mathrm{Mod}_{R_i}^{\mathrm{cn}}, \quad M \mapsto (M \otimes R_i)_{i \in \mathcal{J}}$$

induces a symmetric monoidal functor on p -complete modules

$$F^\wedge : \mathrm{Mod}_R^{\mathrm{cn}, \wedge} \rightarrow \varprojlim_{i \in \mathcal{J}} \mathrm{Mod}_{R_i}^{\mathrm{cn}, \wedge}, \quad M \mapsto ((M \otimes R_i)^\wedge)_{i \in \mathcal{J}}$$

Moreover, if F is an equivalence, then so is $F^\wedge$.

Proof. The first claim is clear from the properties of p -completion discussed above. For the second, note that the functor F always admits a right adjoint which takes a system of R_i -modules M_i to the limit $\lim_i M_i$ computed in the category of connective spectra. Since p -completion commutes with limits, this restricts to an adjunction on the level of p -complete modules and the second claim follows. $\square$

Corollary 4.3. The functor

$$\mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}), \quad R \mapsto \mathrm{Mod}_R^{\mathrm{cn}, \wedge}$$

is cohesive and nilcomplete.

Proof. Apply Lemma 4.2 to Proposition 4.1. $\square$

Finally, we verify that over the p -typical Witt vectors of a perfect $\mathbb{F}_p$ -algebra, the functor $\text{Mod}_{-}^{\text{cn}, \wedge}$ commutes with the p -completion tower.

Proposition 4.4. Let k be a perfect $\mathbb{F}_p$ -algebra and denote by $W(k)$ the p -typical Witt-vectors of k . The functor

$$\text{Mod}_{W(k)}^{\text{cn}, \wedge} \rightarrow \lim_n \text{Mod}_{W(k)/p^n}^{\text{cn}}, \quad N \mapsto N \otimes_{W(k)} W(k)/p^n \quad (2)$$

is a symmetric monoidal equivalence.

Proof. In light of the natural equivalences

$$\begin{aligned} \text{Mod}_{W(k)}^{\text{cn}, \wedge} &\simeq \text{Mod}_{W(k)}(\text{Mod}_{\mathbb{Z}_p}^{\text{cn}, \wedge}) \\ &\simeq \text{Mod}_{W(k)}(\lim_n \text{Mod}_{\mathbb{Z}/p^n}^{\text{cn}}) \\ &\simeq \lim_n \text{Mod}_{W(k)/p^n}(\text{Mod}_{\mathbb{Z}/p^n}^{\text{cn}}) \\ &\simeq \lim_n \text{Mod}_{W(k)/p^n}^{\text{cn}}, \end{aligned}$$

it suffices to consider the case $k = \mathbb{F}_p$ and $W(k) = \mathbb{Z}_p$. The functor (2) admits a right adjoint, which takes $(M_n) \in \varprojlim_n \text{Mod}_{\mathbb{Z}/p^n}^{\text{cn}}$ to the limit $\lim_n M_n$ taken in the category of connective $\mathbb{Z}_p$ -modules. Since p -complete modules are closed under limits, the essential image of this functor is contained in $\text{Mod}_{\mathbb{Z}_p}^{\text{cn}, \wedge}$. Moreover, for any $M \in \text{Mod}_{\mathbb{Z}_p}^{\text{cn}, \wedge}$ we have that

$$\varprojlim_n (M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n) \simeq \varprojlim_n M/p^n \simeq M^\wedge \simeq M.$$

Hence, the counit of the adjunction is an equivalence on p -complete modules.

Conversely, given $(N_k) \in \varprojlim_k \text{Mod}_{\mathbb{Z}/p^k}^{\text{cn}}$ write $N = \lim_k N$ for the limit in connective spectra, equipped with its natural $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$ -module structure. We want to show that, for every n the natural map

$$N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \xrightarrow{\sim} N_n$$

is an equivalence. Since $N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \simeq N/p^n$ and limits are exact, we have an equivalence

$$N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \simeq \lim_{k>n} (N_k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n).$$

Thus, the unit of the adjunction may be written as

$$\lim_{k>n} (N_k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n) \rightarrow \lim_{k>n} (N_k \otimes_{\mathbb{Z}/p^k} \mathbb{Z}/p^n) \simeq N_n$$

and so has fiber given by

$$F_n := \lim_{k>n} (N_k \otimes_{\mathbb{Z}/p^k} \text{fib}(\mathbb{Z}/p^k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^n)).$$

Now we compute the fiber of $\mathbb{Z}/p^k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^n$ as the module

$$\text{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^k, \mathbb{Z}/p^n)[1] \simeq \mathbb{Z}/p^n[1].$$

The reduction map $\mathbb{Z}/p^k \rightarrow \mathbb{Z}/p^{k-1}$ is induced by the map of projective resolutions

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\cdot p^k} & \mathbb{Z}_p \\ \cdot p \downarrow & & \downarrow \text{id} \\ \mathbb{Z}_p & \xrightarrow{\cdot p^{k-1}} & \mathbb{Z}_p, \end{array}$$

hence, on Tor it induces the multiplication by p map

$$\mathbb{Z}/p^n = \text{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^k, \mathbb{Z}/p^n) \xrightarrow{p} \text{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^{k-1}, \mathbb{Z}/p^n) = \mathbb{Z}/p^n.$$

Thus, if we have $k' > k > n$ such that $k' - k > n$, the transition map

$$F_{k'} = N_{k'} \otimes \text{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^k, \mathbb{Z}/p^n) \rightarrow N_k \otimes \text{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^{k-1}, \mathbb{Z}/p^n) = F_k$$

vanishes since the Tor -groups are p^n -torsion. Choosing a cofinal subset $S \subseteq \mathbb{N}_{>n}$ such that $|k' - k| > n$ for any distinct $k', k \in S$, we see that

$$\lim_{k>n} F_k \simeq \lim_{k \in S} F_k \simeq 0$$

vanishes. Thus, since limits are exact, the map $N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \xrightarrow{\sim} N_n$ is an equivalence. $\square$

4.2 Formally étale coalgebras

We now turn our attention more specifically to coalgebras and introduce the class of *formally étale* coalgebras. We begin with the following observation:

Lemma 4.5. Let $\mathcal{C}_-: \text{CAlg}^{\text{cn}} \rightarrow \text{CAlg}(\text{Pr}^{\text{L}})$ be cohesive or nilcomplete. Then the functor $\text{cCAlg}(\mathcal{C}_-): \text{CAlg}^{\text{cn}} \rightarrow \text{Pr}^{\text{L}}$ is also cohesive or nilcomplete, respectively.

Proof. By Lemma 2.6 combined with Lemma 2.8, the functor $\text{cCAlg}(-): \text{CAlg}(\text{Pr}^{\text{L}}) \rightarrow \text{Pr}^{\text{L}}$ commutes with limits. Since both properties are limit conditions, the claim follows. $\square$

In light of this, for the remainder of this section, we put ourselves in the following setting:

Setting 4.6. Fix a cohesive functor $\mathcal{C}_-: \text{CAlg}^{\text{cn}} \rightarrow \text{CAlg}(\text{Pr}^{\text{L}})$ and denote the corresponding coalgebra functor by

$$\text{cA}_- := \text{cCAlg}(\mathcal{C}_-): \text{CAlg}^{\text{cn}} \rightarrow \text{Pr}^{\text{L}},$$

which is again cohesive by Lemma 4.5. Moreover for any $n \geq 0$, Lemma 2.8 implies that forgetting to Cat_{∞} and composing with the mapping space functor $\text{Map}_{\text{Cat}_{\infty}}(\Delta^n, -)$ produces a cohesive functor with values in spaces, which we denote $\text{cA}^{\Delta^n}: \text{CAlg}^{\text{cn}} \rightarrow \mathcal{S}$. Finally, as in Example 2.10, any map $f: R \rightarrow S$ of connective $\mathbb{E}_{\infty}$ -rings induces a coalgebraic adjunction

$$f^*: \text{cA}_R \rightleftarrows \text{cA}_S : f_*,$$

and we refer to f^* as base change and to f_* as Weil restriction along f .

Given a connective $\mathbb{E}_{\infty}$ -ring and any $\psi \in \text{cA}_R^{\Delta^n}$, we want to study the associated moduli problem

$$(\text{cA}_R^{\Delta^n})_{\psi}^-: \text{CAlg}_{/R}^{\text{cn}} \rightarrow \mathcal{S},$$

as in Definition 3.8. By Proposition 3.11, the behavior of this functor on square zero extensions is governed by the associated tangent complex functor

$$T_{\psi}^-: \text{Mod}_R^{\text{cn}} \rightarrow \text{Sp},$$

which, for each $I \in \text{Mod}_R^{\text{cn}}$ has underlying space given by the fiber

$$\Omega^{\infty} T_{\psi}^I \simeq (\text{cA}_R^{\Delta^n})_{\psi}^{R \oplus I},$$

i.e. the space of lifts of ψ along the map $R \oplus I \rightarrow R$. Our first order of business is to, for $n = 0, 1$ describe this space of lifts as a certain mapping space between coalgebras.

To do this, we begin with the following general categorical construction:

Construction 4.7. Suppose we are given categories $\mathcal{E}, \mathcal{D}$ and an adjunction

$$f^* : \mathcal{E} \rightleftarrows \mathcal{D} : f_*$$

together with a retraction $g^* : \mathcal{D} \rightarrow \mathcal{E}$ of f^* . Consider the natural transformation

$$\pi : f_* f^* \xrightarrow{\sim} g^* f^* f_* f^* \rightarrow g^* f^* \xrightarrow{\sim} \text{id}_{\mathcal{E}}$$

defined as the whiskering of the counit $\varepsilon : f^* f_* \rightarrow \text{id}$ as in the diagram:

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{f^*} & \mathcal{D} & \begin{array}{c} \xrightarrow{f^* f_*} \\ \Downarrow \\ \xrightarrow{\text{id}} \end{array} & \mathcal{D} & \xrightarrow{g^*} & \mathcal{E} . \end{array}$$

Unraveling the definitions, we see that for each $B, A \in \mathcal{E}$ the composition

$$\text{Map}_{\mathcal{E}}(B, f_* f^* A) \xrightarrow{\sim} \text{Map}_{\mathcal{D}}(f^* B, f^* A) \xrightarrow{g^*} \text{Map}_{\mathcal{E}}(B, A)$$

takes $\psi : B \rightarrow f_* f^* A$ to the composite $\pi_A \circ \psi$. Thus, for each $\varphi : B \rightarrow A$ we have a natural equivalence between the fiber

$$\text{fib}_{\varphi} \left(\text{Map}_{\mathcal{D}}(f^* B, f^* A) \xrightarrow{g^*} \text{Map}_{\mathcal{E}}(B, A) \right)$$

and the space of dotted lifts in the diagram:

$$\begin{array}{ccc} & & f_* f^* A \\ & \nearrow \text{dotted} & \downarrow \pi_A \\ B & \xrightarrow{\varphi} & A \end{array}$$

i.e. the mapping space $\text{Map}_{\mathcal{E}/A}((B \xrightarrow{\varphi} A), (f_* f^* A \xrightarrow{\pi_A} A))$.

This allows us to make our first crucial definition specific to the deformation theory of coalgebras.

Definition 4.8. Let R be a connective $\mathbb{E}_{\infty}$ -ring and I a connective R -module. Denote by $s_I : R \rightarrow R \oplus I$ the 0-section of the split square zero extension. For any $A \in \mathcal{cA}_R$ we define the *universal deformation coalgebra* of the pair (A, I) as the Weil restriction of the base change:

$$\mathfrak{D}_A(I) := (s_I)_* (s_I)^* A \in \mathcal{cA}_R.$$

This coalgebra comes equipped with two natural maps

$$A \xrightarrow{\varepsilon_I} \mathfrak{D}_A(I) \xrightarrow{\pi_I} A \in \mathcal{cA}_R,$$

where the first is the unit of the adjunction $s_I^* \dashv (s_I)_*$ and the second is the map of Construction 4.7, applied to the retraction of s_I^* provided by base change along the projection $R \oplus I \rightarrow R$. Moreover, for any map $\varphi : B \rightarrow A \in \mathcal{cA}_R$, we call

$$\text{Der}_{\varphi}(B, I) := \text{Map}_{(\mathcal{cA}_R)/A}(B, \varphi), (\mathfrak{D}_A(I), \pi_A))$$

the space of *derivations* from B to I .

Derivations with respect to the identity morphism naturally correspond to solutions of certain lifting problems as follows:

Lemma 4.9. For any connective $\mathbb{E}_{\infty}$ -ring R , $I \in \text{Mod}_R^{\text{cn}}$ and $A \in \mathcal{cA}_R$ we have a natural equivalence

$$(\mathcal{cA}_R^{\Delta^0})_A^{R \oplus I} \simeq \text{Der}_{\text{id}_A}(A, \Sigma I).$$

Proof. Let $s_{\Sigma I}: R \rightarrow R \oplus \Sigma I$ denote the 0-section of the split square zero extension. Observe that, since $\mathcal{cA}_-^{\Delta^0}$ is cohesive and fibers commute with forming the loop space, we have natural equivalences

$$(\mathcal{cA}_R^{\Delta^0})_A^{R \oplus I} \simeq \Omega(\mathcal{cA}_R^{\Delta^0})_A^{R \oplus \Sigma I} \simeq \text{fib}_{\text{id}_A} \left(\text{Map}_{\mathcal{cA}_{R \oplus \Sigma I}}(s_{\Sigma I}^* A, s_{\Sigma I}^* A) \rightarrow \text{Map}_{\mathcal{cA}_R}(A, A) \right).$$

Moreover, this fiber is naturally identified with $\text{Der}_{\text{id}_A}(A, \Sigma I)$, by definition of the space of derivations and the discussion in Construction 4.7. $\square$

The following Lemma shows that we can also describe the deformation space of a map of coalgebras in terms of derivations.

Lemma 4.10. Let R an $\mathbb{E}_\infty$ -ring and $\varphi: B \rightarrow A \in \mathcal{cA}_R$ be a morphism of coalgebras. Then for every connective R -module I , the space $(\mathcal{cA}_R^{\Delta^1})_\varphi^{R \oplus I}$ is given by the pullback

$$(\mathcal{cA}_R^{\Delta^1})_\varphi^{R \oplus I} \simeq \text{Der}_{\text{id}}(B, \Sigma I) \times_{\text{Der}_\varphi(B, \Sigma I)} \text{Der}_{\text{id}}(A, \Sigma I).$$

Proof. Write $s_{\Sigma I}: R \rightarrow R \oplus \Sigma I$ for the 0-section of the split square zero extension. If $\mathcal{E}$ is any category and $(\psi: X \rightarrow Y) \in \mathcal{E}^{\Delta^1}$ some morphism, then the loop space at ψ sits in a pullback diagram of the form:

$$\begin{array}{ccc} \Omega_\psi \mathcal{E}^{\Delta^1} & \longrightarrow & \text{Aut}_\mathcal{E}(X) \\ \downarrow & & \downarrow \psi \circ - \\ \text{Aut}_\mathcal{E}(Y) & \xrightarrow{- \circ \psi} & \text{Map}_\mathcal{E}(X, Y). \end{array}$$

Applying this to the category $\mathcal{E} = \mathcal{cA}_{R \oplus \Sigma I}$ and the morphism $\psi = s_{\Sigma I}^* \varphi: s_{\Sigma I}^* B \rightarrow s_{\Sigma I}^* A$ and taking the respective fibers of the base change along the map $R \oplus I \rightarrow R$, we obtain a pullback square:

$$\begin{array}{ccc} \Omega_{s_{\Sigma I}^* \varphi} (\mathcal{cA}_R^{\Delta^1})_\varphi^{R \oplus \Sigma I} & \longrightarrow & \text{Der}_{\text{id}_B}(B, \Sigma I) \\ \downarrow & & \downarrow \\ \text{Der}_{\text{id}_A}(A, \Sigma I) & \longrightarrow & \text{Der}_\varphi(B, \Sigma I). \end{array}$$

Since $\mathcal{cA}_-^{\Delta^1}$ is cohesive, the upper left corner is naturally equivalent to $(\mathcal{cA}_R^{\Delta^1})_\varphi^{R \oplus I}$ and so we conclude. $\square$

Before we proceed, we investigate the functoriality of the universal deformation coalgebra in the input module. For this, we first need a general categorical lemma.

Lemma 4.11. Let $R_{(-)}: \mathcal{J} \rightarrow \text{CAlg}^{\text{cn}}$ be a diagram with limit R , such that the natural map $\mathcal{cA}_R \rightarrow \lim_i \mathcal{cA}_{R_i}$ is an equivalence of categories. Moreover, suppose we are given a map $f: S \rightarrow R$ with projections $f_i: S \rightarrow R_i$. Then for any $A \in \mathcal{cA}_S$ we have a natural equivalence

$$f_* f^* A \simeq \lim_i (f_i)_* (f_i)^* A \in \mathcal{cA}_S.$$

Proof. Unraveling the adjunctions and using the Yoneda lemma, this is equivalent to the claim that for any $B \in \mathcal{cA}_S$ the natural map

$$\text{Map}_{\mathcal{cA}_R}(f^* B, f^* A) \rightarrow \lim_i \text{Map}_{\mathcal{cA}_{R_i}}(f_i^* B, f_i^* A)$$

is an equivalence, which is precisely the formula for mapping spaces in the limit $\mathcal{cA}_R = \lim_i \mathcal{cA}_{R_i}$. $\square$

With this in hand, we can show that the universal deformation coalgebra naturally refines to a spectrum object in the category of coalgebras.

Proposition 4.12. Let R be a connective $\mathbb{E}_\infty$ -ring and $A \in \mathcal{cA}_R$. The assignment taking $I \in \text{Mod}_R^{\text{cn}}$ to the augmented coalgebra $\mathfrak{D}_A(I)$ refines to an excisive functor

$$L_A: \text{Mod}_R^{\text{cn}} \rightarrow \text{Sp}((\mathcal{cA}_R)_{A/A}).$$

Proof. Indeed, since $\mathcal{cA}_- = \mathcal{cAlg}(\mathcal{C}_-)$ is cohesive, we have a pullback diagram in Pr^{L} of the form

$$\begin{array}{ccc} \mathcal{cA}_{R \oplus I} & \longrightarrow & \mathcal{cA}_R \\ \downarrow & & \downarrow \\ \mathcal{cA}_R & \longrightarrow & \mathcal{cA}_{R \oplus \Sigma I}, \end{array}$$

where the maps are base change along the projection $p_I: R \oplus I \rightarrow R$ and 0-section $s_{\Sigma I}: R \rightarrow R \oplus \Sigma I$ respectively. Moreover, base change along the 0-section $s_I: R \rightarrow R \oplus I$ provides a map from $\mathcal{cA}_R$ into the pullback $\mathcal{cA}_{R \oplus I}$. Unwrapping the definitions and using that $p_I \circ s_I \simeq \text{id}$, Lemma 4.11 produces for us a pullback square in $\mathcal{cA}_R$ of the form

$$\begin{array}{ccc} \mathfrak{D}_A(I) & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathfrak{D}_A(\Sigma I), \end{array}$$

where the top and left hand maps are those of Construction 4.7 and the bottom and right hand maps are the unit of the base change adjunction $s_{\Sigma I}: R \rightarrow R \oplus \Sigma I$. Thus, the claim follows by combining [Lur17, Proposition 1.4.2.13] and [Lur17, Proposition 1.4.2.22]. $\square$

Definition 4.13. Let R be a connective $\mathbb{E}_\infty$ -ring, $A \in \mathcal{cA}_R$ and denote by $L_A: \text{Mod}_R^{\text{cn}} \rightarrow \text{Sp}(\mathcal{cA}_R)_{A/A}$ the functor of Proposition 4.12. We say that A is *formally étale* if $L_A(R) \simeq 0$ and denote by

$$\mathcal{cA}_R^{\text{fét}} \subseteq \mathcal{cA}_R$$

the full subcategory spanned by the formally étale coalgebras.

Lemma 4.14. For a connective $\mathbb{E}_\infty$ -ring R and any coalgebra $A \in \mathcal{cA}_R$ the following are equivalent:

1. A is formally étale.
2. For any $n \geq 0$ either of the natural maps $A \xrightarrow{\varepsilon} \mathfrak{D}_A(\Sigma^n R) \xrightarrow{\pi} A$ is an equivalence.
3. The functor L_A is null when restricted to the full subcategory of perfect R -modules.
4. For any perfect connective R -module I , either of the natural maps $A \xrightarrow{\varepsilon} \mathfrak{D}_A(I) \xrightarrow{\pi} A$ is an equivalence.
5. For any perfect connective R -module I and any map of coalgebras $\varphi: B \rightarrow A$, the space $\text{Der}_\varphi(B, I)$ is contractible.

Proof. It is clear from the definitions that the equivalences $1. \Leftrightarrow 2.$ as well as $3. \Leftrightarrow 4. \Leftrightarrow 5.$ hold. Moreover, $3. \Rightarrow 1.$ since R itself a perfect R -module, so it suffices to show $1. \Rightarrow 3.$. Since $L_A(-)$ is excisive by Proposition 4.12, this follows from the fact that any perfect R -module admits a finite filtration by free R -modules and so we conclude. $\square$

Remark 4.15. The following closure properties of formally étale coalgebras are immediate from the definitions:

1. Since the product of coalgebras is given by the underlying tensor product and base change is symmetric monoidal, finite products of formally étale coalgebras are formally étale.

2. Retracts of formally étale coalgebras are formally étale.
3. If $\mathcal{C}_-$ has the property that for any $n \geq 0$ base change along the zero section $R \rightarrow R\{\varepsilon_n\}$ admits a left adjoint, then formally étale coalgebras are closed under limits⁵. Note that, this is satisfied for the moduli functor $R \mapsto \text{Mod}_R^{\text{cn}}$ as well as its p -complete variant, as $R\{\varepsilon_n\}$ is a perfect R -module.

We will also need the following, more technical characterization of formally étale coalgebras:

Proposition 4.16. Let R be a connective $\mathbb{E}_\infty$ -ring and $A \in \mathcal{CA}_R$. For any map of coalgebras $\varphi: B \rightarrow A \in \mathcal{CA}_R$ and any $I \in \text{Mod}_R^{\text{cn}}$, consider the map

$$T_\varphi^I \rightarrow T_B^I \in \text{Mod}_R, \quad (3)$$

induced by evaluating at the domain. The following statements are equivalent:

1. A is formally étale
2. The map (3) is an equivalence for all $\varphi: B \rightarrow A$ and $I = R$
3. The map (3) is an equivalence for all $\varphi: B \rightarrow A$ and all perfect connective R -modules I .

Proof. First, note that 2. $\Leftrightarrow$ 3. holds by Proposition 3.12, so it suffices to show that 3. $\Leftrightarrow$ 1.. By construction of tangent spectrum, the map (3) is an equivalence for all triples (B, φ, I) as in 3. if and only if the map on the underlying fibers

$$(\mathcal{CA}_R^{\Delta^1})_\varphi^{R \oplus I} \rightarrow (\mathcal{CA}_R^{\Delta^0})_B^{R \oplus I} \quad (4)$$

is an equivalence for such triples. Given some lift $B' \in (\mathcal{CA}_R^{\Delta^0})_B^{R \oplus I}$ of B along the map $R \oplus I \rightarrow R$, write

$$F_{\varphi, B'}^I := \text{fib}_{B'} \left((\mathcal{CA}_R^{\Delta^1})_\varphi^{R \oplus I} \rightarrow (\mathcal{CA}_R^{\Delta^0})_B^{R \oplus I} \right)$$

for the fiber of B' under the map (4). Using the identification $\text{Der}_{\text{id}}(B, \Sigma I) \simeq (\mathcal{CA}_R^{\Delta^0})_B^{R \oplus I}$ of Lemma 4.9, we learn from Lemma 4.10 that we have two pullback squares:

$$\begin{array}{ccccc} F_{\varphi, B'}^I & \longrightarrow & (\mathcal{CA}_R^{\Delta^1})_\varphi^{R \oplus I} & \longrightarrow & \text{Der}_{\text{id}}(A, \Sigma I) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \text{Der}_{\text{id}}(B, \Sigma I) & \longrightarrow & \text{Der}_\varphi(B, \Sigma I). \end{array}$$

Thus, by the pasting law for pullbacks, we obtain a further fiber sequence

$$F_{\varphi, B'}^I \rightarrow \text{Der}_{\text{id}}(A, \Sigma I) \xrightarrow{-\circ \varphi} \text{Der}_\varphi(B, \Sigma I).$$

If A is formally étale, the middle and right hand term in this sequence are contractible, and hence so is the left hand term. In particular, the map (4) is an equivalence, meaning the “only if” direction holds.

Conversely, assume that $F_{\varphi, B'}^I$ is contractible for every triple (φ, B, I) as above and every lift $B' \in (\mathcal{CA}_R^{\Delta^0})_B^{R \oplus I} \simeq \text{Der}_{\text{id}}(B, \Sigma I)$. Applying this to the unique map $0: \emptyset \rightarrow A$ from the initial coalgebra $\emptyset \in \mathcal{CA}_R$ we see that

$$\text{Der}_{\text{id}}(A, \Sigma I) \xrightarrow{\sim} \text{Der}_0(\emptyset, \Sigma I) \simeq *.$$

Applying the same logic to an arbitrary triple (φ, B, I) we get

$$\text{Der}_\varphi(B, I) \simeq \Omega \text{Der}_\varphi(B, \Sigma I) \simeq *$$

which implies that A was formally étale by Lemma 4.14, meaning the “if” direction holds, as claimed. $\square$

⁵As the left adjoint will be oplax monoidal, and hence induce a left adjoint to base change on the level of coalgebras

With this in hand, we can show that lifts of formally étale coalgebras along nilpotent thickenings are again formally étale.

Proposition 4.17. Let $f: R' \rightarrow R$ be a nilpotent thickening of connective $\mathbb{E}_\infty$ -rings and let $A \in \mathcal{CA}_{R'}$. If the base change f^*A is formally étale, then so is A .

Proof. Arguing iteratively, we can assume that $R' \rightarrow R$ is a square zero extension with perfect fiber I . Then by Proposition 4.16 it suffices to show that, for any morphism of coalgebras $\varphi: B \rightarrow A \in \mathcal{CA}_{R'}$ the natural map $T_\varphi^{R'} \rightarrow T_A^{R'}$ is an equivalence. By assumption, we have a fiber sequence of R' -modules

$$I \rightarrow R' \rightarrow R,$$

where the R' -action on the outer terms factors through R . Applying the natural transformation $\text{ev}_0: T_\varphi^- \rightarrow T_B^-$ to this fiber sequence and using that the tangent spectrum functors are excisive, we obtain a commutative diagram

$$\begin{array}{ccccccc} T_{f^*\varphi}^I & \xleftarrow{\sim} & T_\varphi^I & \longrightarrow & T_\varphi^{R'} & \longrightarrow & T_\varphi^R \xrightarrow{\sim} T_{f^*\varphi}^R \\ \downarrow \sim & & \downarrow & & \downarrow & & \downarrow \sim \\ T_{f^*B}^I & \xleftarrow{\sim} & T_B^I & \longrightarrow & T_B^{R'} & \longrightarrow & T_B^R \xrightarrow{\sim} T_{f^*B}^R \end{array}$$

where the inner two horizontal maps in each row form a cofiber sequence. The outer horizontal maps are the base change equivalences from Proposition 3.13 and the outer vertical maps are equivalences since f^*A is formally étale by assumption. Thus, the middle map $T_\varphi \rightarrow T_B$ is an equivalence as well and we conclude. $\square$

We can neatly organize the results of this section into the following proposition:

Proposition 4.18. Let $f: R' \rightarrow R$ be a nilpotent thickening of connective $\mathbb{E}_\infty$ -rings. Then Weil-restriction along f defines a fully faithful functor

$$f_*: \mathcal{CA}_R^{\text{fét}} \rightarrow \mathcal{CA}_{R'}^{\text{fét}}.$$

Moreover, for any $A \in \mathcal{CA}_R^{\text{fét}}$ the coalgebra f_*A is, up to contractible choice, the unique lift of A to $\mathcal{CA}_{R'}$.

Proof. Using Proposition 4.17, we may argue iteratively and reduce to the case where the map $f: R' \rightarrow R$ is a square zero extension with fiber given by a perfect connective R -module I . Since A is formally étale, we have that $\text{Der}_{\text{id}_A}(A, \Sigma I)$ is contractible and hence the identification of Lemma 4.10 combined with Proposition 3.11 implies that the space of lifts is non-empty and contractible. Denote by $A' \in \mathcal{CA}_{R'}$ the unique lift, which is formally étale by Proposition 4.17.

Moreover, for any $B \in \mathcal{CA}_{R'}$ and any map of coalgebras $\varphi: f^*B \rightarrow A$, similarly combining 4.16 with Proposition 3.11 shows that the natural map

$$\text{Map}_{\mathcal{CA}_{R'}}(B, A') \xrightarrow{\sim} \text{Map}_{\mathcal{CA}_R}(f^*B, A)$$

is an equivalence, and so $A' = f_*A$ by Yoneda. Moreover, as A' was a lift of A , the counit $f^*f_*A \xrightarrow{\sim} A$ is an equivalence, which implies that the restriction of f_* to formally étale coalgebras is fully faithful, as claimed. $\square$

Under the natural completeness assumption on the underlying moduli $\mathcal{C}_-$, we may similarly lift against pro-nilpotent thickenings. This will be crucial for us to pass from $\mathbb{F}_p$ to the p -completed sphere $\mathbb{S}_p^\wedge$ in the next section.

Proposition 4.19. Let $f = \lim_i f_i: R = \lim_i R_i \rightarrow R_0$ be a direct limit of nilpotent thickenings, such that the natural functor $\mathcal{C}_R \rightarrow \lim_i \mathcal{C}_{R_i}$ is an equivalence of categories. Then Weil restriction along f restricts to a fully faithful functor

$$f_*: \mathcal{CA}_{R_0}^{\text{fét}} \rightarrow \mathcal{CA}_R^{\text{fét}},$$

such that for any $A_0 \in \mathcal{CA}_{R_0}^{\text{fét}}$ the coalgebra f_*A_0 is, up to contractible choice, the unique lift of A_0 to an object of $\mathcal{CA}_R$.

Proof. First, note that by Lemma 2.6 the map $\mathcal{CA}_R \rightarrow \varprojlim_i \mathcal{CA}_{R_i}$ is also an equivalence of categories. Now for any $A \in \mathcal{CA}_R$ such that $A_0 = f^*A$ is formally étale, Proposition 4.17 implies that each base change $A_i = f_i^*A$ is formally étale as well. Using this and the fact that fibers commute with limits, as well as the description of the tangent spectra from Lemma 4.9, it is straightforward to verify that the condition of Proposition 4.16 is satisfied and thus A was itself formally étale.

Moreover, Weil restriction along any of the finite stages $R_i \rightarrow R_j$ is fully faithful on formally étale coalgebras by Proposition 4.18. Since mapping spaces in a limit of categories are given by the limit of mapping spaces, any limit of fully faithful functors is fully faithful and so f_* is fully faithful when restricted to formally étale coalgebras as claimed. The fact that f_*A is the unique lift of A is similarly immediate by Proposition 4.18 and the fact that fibers commute with limits. $\square$

Let us now discuss some examples of formally étale coalgebras. For any coalgebra $A \in \text{cCAlg}(\mathcal{C})$ an *idempotent* of A is a linear map $e: A \rightarrow \mathbb{1}_{\mathcal{C}}$ satisfying $e^2 = e$, i.e. an idempotent of the algebra $\pi_0 A^\vee$. Recall from [BB24, Section 6] that, given such an idempotent, the localization

$$A[e^{-1}] = \lim \left(A \xrightarrow{e} A \xrightarrow{e} \cdots \right) \in \text{cCAlg}(\mathcal{C})$$

exists and gives a decomposition of coalgebras

$$A[e^{-1}] \oplus A[(e - 1)^{-1}] \xrightarrow{\sim} A$$

such that the component maps are monomorphisms in $\text{cCAlg}(\mathcal{C})$ and $\mathcal{C}$.

Lemma 4.20. Let $\mathcal{C}$ be a symmetric monoidal category which admits arbitrary coproducts and is compactly generated by the unit $\mathbb{1}_{\mathcal{C}}$. For any set S and any $s \in S$, write $\pi_s: \mathbb{1}_{\mathcal{C}}[S] \rightarrow \mathbb{1}_{\mathcal{C}}$ for the idempotent given by projecting to the s -factor. Any map of coalgebras $f: A \rightarrow \mathbb{1}_{\mathcal{C}}[S]$ induces an equivalence

$$c_f: \prod_{s \in S} A[(\pi_s f)^{-1}] \xrightarrow{\sim} A \in \text{cCAlg}(\mathcal{C}).$$

Moreover, this construction refines to an equivalence of categories

$$\text{cCAlg}(\mathcal{C})_{/\mathbb{1}_{\mathcal{C}}[S]} \xrightarrow{\sim} \prod_S \text{cCAlg}(\mathcal{C}).$$

Proof. Note that c_f is a monomorphism in $\mathcal{C}$ and hence it suffices to check that it is surjective on π_0 . Given any $v \in \pi_0 A$, its image under the map $\pi_0 f: \pi_0 A \rightarrow \pi_0 \mathbb{1}_{\mathcal{C}}[S]$ must be supported on some finite $S' \subseteq S$. In particular, we reduce to the case of finitely many orthogonal idempotents, and so v is in the image of c_f . The second claim now follows from the fact that the maps $A[(\pi_s f)^{-1}] \rightarrow A$ are orthogonal localizations, by construction. $\square$

Corollary 4.21. For any $\mathbb{E}_\infty$ -ring R and any set S , the coalgebra $\mathbb{1}_{\mathcal{C}_R}[S]$ is formally étale.

Proof. This follows readily from Lemma 4.20, as the coalgebra $\mathbb{1}_{\mathcal{C}_R}[S]$ satisfies the same universal property as the base change to any split square zero extension. $\square$

For dualizable coalgebras, our notions of formally étale and derivations are compatible with the usual notions from higher algebra.

Proposition 4.22. Let R be a connective $\mathbb{E}_\infty$ -ring and let $A \in \text{cCAlg}(\text{Mod}_R^{\text{cn}})$ be dualizable. For every map $\varphi: B \rightarrow A \in \text{cCAlg}(\text{Mod}_R^{\text{cn}})$ and every $I \in \text{Mod}_R^{\text{cn}}$, we have an equivalence

$$\text{Der}_\varphi(B, I) \simeq \text{Map}_{A^\vee}(L_{A^\vee/R}, \text{map}_R(B, I)),$$

where $L_{A^\vee/R}$ denotes the relative cotangent complex as in Example 3.15.

Proof. Set $R' = R \oplus I$ and denote by $s_I: R \rightarrow R'$ the 0-section. By construction, the space $\text{Der}_\varphi(B, I)$ is equivalent to the fiber

$$F_\varphi := \text{fib}_\varphi \left(\text{Map}_{\text{cCAlg}(\text{Mod}_{R'})}(s_I^* B, s_I^* A) \rightarrow \text{Map}_{\text{cCAlg}(\text{Mod}_R)}(B, A) \right)$$

Since A is dualizable, so is $s_I^* A$ with dual given by $(s_I^* A)^\vee \simeq A^\vee \otimes_R R'$. Using Corollary 2.14 and pushing around adjunctions in an analogous manner to Construction 4.7, we learn that F_φ is given by the space of lifts in the diagram

$$\begin{array}{ccc} & \text{map}_R(B, R') & \\ & \downarrow & \\ A^\vee & \xrightarrow{\quad} & B^\vee, \end{array}$$

in $\text{CAlg}(\text{Mod}_R)$. Since $R' \rightarrow R$ is a split square zero extension with fiber I , the morphism $\text{map}_R(B, R') \rightarrow B^\vee$ is a square zero extension as well with fiber given by the $B^\vee$ -module $\text{map}_R(B, I)$ ⁶. Hence, we get

$$F_\varphi \simeq \text{Map}_{A^\vee}(L_{A^\vee/R}, \text{map}_R(B, I))$$

by construction of the relative cotangent complex, as claimed. $\square$

This provides us with the most easily accessible examples of formally étale coalgebras, namely those dual to formally étale algebras.

Corollary 4.23. Let R be a connective $\mathbb{E}_\infty$ -ring. If $A \in \text{cCAlg}(\text{Mod}_R^{\text{cn}})$ is dualizable and the relative cotangent complex $L_{A^\vee/R}$ vanishes, then A is formally étale.

Although we will not make explicit use of it in the next section, we note that the above Corollary applies to the $\mathbb{F}_p$ -homology of compact spaces.

Example 4.24. For any compact space $X \in S^\omega$ the $\mathbb{F}_p$ -homology $\mathbb{F}_p[X]$ is compact and hence dualizable in $\text{Mod}_{\mathbb{F}_p}$ with dual given by the cohomology $\mathbb{F}_p^X$. By [Lur11, Proposition 2.4.12] we have $L_{\mathbb{F}_p^X/\mathbb{F}_p} \simeq 0$ and so Corollary 4.23 tells us that $\mathbb{F}_p[X]$ is a formally étale coalgebra.

Remark 4.25. It is unclear whether the converse of Corollary 4.23 holds. From Proposition 4.22 we can only deduce that

$$\text{Map}_{A^\vee}(L_{A^\vee/R}, \varphi_*^\vee \text{map}_R(B, I)) \simeq 0$$

for each coalgebra B , R -module I and morphism of algebras $\varphi: A^\vee \rightarrow B^\vee$.

5 Homology coalgebras

As observed in Example 2.16, for every space X and every $\mathbb{E}_\infty$ -ring R , the R -homology $R[X]$ carries a natural R -coalgebra structure induced from the diagonal map $\Delta: X \rightarrow X \times X$. The main result of this section is Theorem 5.7, namely that for any space X and any perfect field k of characteristic $p > 0$, the coalgebra $k[X]$ is formally étale.

Theorem 5.1. Let p be a prime, k a perfect $\mathbb{F}_p$ -algebra with spherical Witt-vectors $\mathbb{W}(k)$. Then Weil restriction along the natural map $\mathbb{W}(k) \rightarrow k$ induces a fully faithful functor

$$\mathcal{R}_{\mathbb{W}(k)/k}: \text{cCAlg}(\text{Mod}_k^{\text{cn}})^{\text{fét}} \rightarrow \text{cCAlg}(\text{Mod}_{\mathbb{W}(k)}^{\text{cn}, \wedge})^{\text{fét}}$$

such that, for any $A \in \text{cCAlg}(\text{Mod}_k^{\text{cn}})^{\text{fét}}$ the image $\mathcal{R}_{\mathbb{W}(k)/k}(A)$ is, up to contractible choice, the unique p -complete, connective $\mathbb{W}(k)$ -coalgebra equipped with a natural equivalence $\mathcal{R}_{\mathbb{W}(k)/k}(A) \otimes k \simeq A$.

⁶Where we again the lax monoidality of the functor $\text{map}_R(-, -)$

Proof. First of all, recall that by Corollary 4.3, the functor $R \mapsto \text{Mod}_R^{\text{cn}, \wedge}$ is cohesive, so we can talk about formally étale coalgebras in the sense of Definition 4.13. Since k is discrete, our map in question factors as $\mathbb{W}(k) \rightarrow \pi_0 \mathbb{W}(k) \rightarrow k$, where $\pi_0 \mathbb{W}(k)$ naturally identifies with the ordinary p -typical Witt vectors of k . Since the first map is a pro-nilpotent thickening via the Postnikov tower and the second via the p -completion tower, the claim follows by using Corollary 4.3 and Proposition 4.4 to apply Proposition 4.19 twice. $\square$

Let k be a field of characteristic $p > 0$ and write $C_k: \text{Mod}_k^{\text{cn}} \rightarrow \text{cAlg}(\text{Mod}_k^{\text{cn}})$ for the connective cofree coalgebra functor as in Definition 2.5. For any $n \geq 0$, the coalgebra $C_k(\Sigma^n k)$ carries a Frobenius endomorphism F , which parametrizes the stable power operation

$$P^0: \Omega^{\infty-n} A^\vee \rightarrow \Omega^{\infty-n} A^\vee$$

acting on the dual algebra $A^\vee$ of any k -coalgebra A . Bachmann–Burklund [BB24, Lemma 7.8] show that for any $n \geq 1$, we have a natural pullback square of connective⁷ coalgebras

$$\begin{array}{ccc} k[\Omega^\infty \Sigma^n \mathbb{F}_p] & \longrightarrow & C_k(\Sigma^n k) \\ \downarrow & & \downarrow \text{id} - F \\ k & \longrightarrow & C_k(\Sigma^n k). \end{array}$$

Now let $f: k \rightarrow R$ be a map of connective $\mathbb{E}_\infty$ -rings such that the underlying k -module of R is dualizable. Then the base change functor $f^*: \text{Mod}_k^{\text{cn}} \rightarrow \text{Mod}_R^{\text{cn}}$ admits an oplax monoidal left adjoint and so the induced functor $f^*: \text{cAlg}(\text{Mod}_k^{\text{cn}}) \rightarrow \text{cAlg}(\text{Mod}_R^{\text{cn}})$ preserves all limits and commutes with cofree coalgebras. Thus, via base change we obtain a further pullback diagram:

$$\begin{array}{ccc} R[\Omega^\infty \Sigma^n \mathbb{F}_p] & \longrightarrow & C_R(\Sigma^n R) \\ \downarrow & & \downarrow f^*(\text{id} - F) \\ R & \longrightarrow & C_R(\Sigma^n R). \end{array} \tag{5}$$

which we call the R -linear *Artin–Schreier square*.

For any $i \geq 0$, recall that we write $k\{\epsilon_i\} = k \oplus \Sigma^i k$ for the split square zero extension on a generator in topological degree i . The key observation is the following:

Lemma 5.2. Let $i \geq 0$ and write $s_i: k \rightarrow k\{\epsilon_i\}$ for the zero section of the split square zero extension. The Weil-restriction of the base-changed Artin–Schreier map

$$(s_i)_*(s_i)^*(\text{id} - F): C_k(\Sigma^n k\{\epsilon_i\}) \rightarrow C_k(\Sigma^n k\{\epsilon_i\})$$

naturally splits in $\text{cAlg}(\text{Mod}_k^{\text{cn}})$ as

$$(\text{id} - F) \otimes_k \text{id}: C_k(\Sigma^n k) \otimes C_k(\Sigma^{n+i} k) \rightarrow C_k(\Sigma^n k) \otimes C_k(\Sigma^{n+i} k).$$

Proof. Unwinding the definitions, we see that the map $(s_i)_*(s_i)^*(\text{id} - F)$ represents the additive operation $\text{id} - P^0$ acting on the space $\Omega^{\infty-n}(A^\vee \oplus \Sigma^i A^\vee)$, induced by the split square zero structure on the dual algebra $(A \otimes k\{\epsilon_i\})^\vee \simeq A^\vee \oplus \Sigma^i A^\vee$. Since the operation P^0 is given by extracting the coefficients of the extended p -th power map, the discussion in Remark 3.2 implies that P^0 acts trivially on the square zero part and so the claim follows. $\square$

Proposition 5.3. For any $\mathbb{F}_p$ -vector space $V \in \text{Mod}_{\mathbb{F}_p}^\heartsuit$ and any $n \geq 1$, the coalgebra $k[\Omega^\infty \Sigma^n V] \in \text{cAlg}(\text{Mod}_k^{\text{cn}})$ is formally étale.

⁷The result in [BB24] is phrased as a pullback in the category of all coalgebras, but we obtain the square below by applying the (limit preserving) connective cover functor of Example 2.9.

Proof. Choosing bases, we may write V as a retract of a (constant) product $\prod_S \mathbb{F}_p$ where S is some set. Since any functor preserves retracts, this exhibits $k[\Omega^\infty \Sigma^n V]$ as a retract of the coalgebra

$$k[\Omega^\infty \prod_S \Sigma^n \mathbb{F}_p] \simeq k[S \times \Omega^\infty \Sigma^n \mathbb{F}_p] \simeq k[S] \otimes k[\Omega^\infty \Sigma^n \mathbb{F}_p].$$

Since $k[S]$ is formally étale by Corollary 4.21 and formally étale coalgebras are closed under retracts and tensor products, this lets us reduce to the case $V = \mathbb{F}_p$.

Thus, for any $i \geq 0$, consider the associated Artin–Schreier square (5) in the case of the split square zero extension $R = k\{\epsilon_i\}$ by a generator in topological degree i . Weil-restricting along the zero-section $s_i: k \rightarrow k\{\epsilon_i\}$, Lemma 5.2 tells us that we have a pullback square in $\text{cAlg}(\text{Mod}_k^{\text{cn}})$ of the form:

$$\begin{array}{ccc} (s_i)_*(s_i)^*k[\Omega^\infty \Sigma^n \mathbb{F}_p] & \longrightarrow & C_k(\Sigma^n k) \otimes C_k(\Sigma^{n+i} k) \\ \downarrow & & \downarrow (\text{id} - F) \otimes \text{id} \\ k & \longrightarrow & C_k(\Sigma^n k) \otimes C_k(\Sigma^{n+i} k). \end{array}$$

Since the product on the right hand splits off the identity map, this diagram is equivalent to the k -linear Artin–Schreier square and we learn that the natural map

$$k[\Omega^\infty \Sigma^n \mathbb{F}_p] \rightarrow (s_i)_*(s_i)^*k[\Omega^\infty \Sigma^n \mathbb{F}_p]$$

is in fact an equivalence. Thus, the coalgebra $k[\Omega^\infty \Sigma^n \mathbb{F}_p]$ is formally étale, and we conclude. $\square$

We now quickly recall some facts about pro-objects and refer the reader to [BB24, Section 5] for a somewhat more in-depth discussion. Let $\mathcal{C}$ be a stable symmetric monoidal category equipped with a compatible t -structure. We denote by $\text{Pro}(\mathcal{C})$ the category of pro-objects in $\mathcal{C}$. We think of objects of $\text{Pro}(\mathcal{C})$ as formal cofiltered limits of objects of $\mathcal{C}$. If $\mathcal{C}$ admits all small limits, the restricted Yoneda embedding gives an adjunction

$$c: \mathcal{C} \rightleftarrows \text{Pro}(\mathcal{C}) : M$$

Here we think of c (aka restricted Yoneda) as taking $X \in \mathcal{C}$ to the constant diagram on X . The functor M is called *materialization* and takes a formal limit $\varprojlim_\lambda X_\lambda$ to the actual limit computed in $\mathcal{C}$. The category $\text{Pro}(\mathcal{C})$ inherits a natural symmetric monoidal structure from $\mathcal{C}$ via the pointwise tensor product, such that the inclusion $\mathcal{C} \rightarrow \text{Pro}(\mathcal{C})$ natural refines to a monoidal functor.

We call an object $\varprojlim_\lambda X_\lambda \in \text{Pro}(\mathcal{C})$ *pro-truncated* if each X_λ is bounded above. The inclusion of pro-truncated objects admits a left adjoint denoted $\tau_{<\infty}$, which can explicitly described by the formula

$$\tau_{<\infty} \varprojlim_\lambda X_\lambda = \lim_{n,\lambda} \tau_{\leq n} X_\lambda.$$

An object $E \in \text{Pro}(\mathcal{C})$ is called *pro-constant up to pro-truncation* if the natural map

$$\tau_{<\infty} cM(E) \rightarrow \tau_{<\infty} E$$

is an equivalence in $\text{Pro}(\mathcal{C})$. If $\mathcal{C}$ has the property that bounded above objects are closed under tensor product, the pro-truncation functor $\tau_{<\infty}$ inherits a natural oplax monoidal structure, and thus induce a functor on coalgebras which we denote the same.

We denote the partially defined right adjoint to $c: \text{cAlg}(\mathcal{C}) \rightarrow \text{cAlg}(\text{Pro}(\mathcal{C}))$ by M^{cA} and refer to it as *coalgebraic materialization*.

Lemma 5.4. Let X be a pointed, connected space and k a field of characteristic p . Then the coalgebra $k[\Omega^\infty \mathbb{F}_p \otimes \Sigma^\infty X]$ is given by a limit of coalgebras $k[Y_n]$, where each Y_n is a finite product of connected Eilenberg–MacLane spaces.

Proof. Set $Y = \Omega^\infty \mathbb{F}_p \otimes \Sigma^\infty X$ and $V_i = \pi_i(\mathbb{F}_p \otimes \Sigma^\infty X)$. Since $\mathbb{F}_p$ is a field, we have that

$$\mathbb{F}_p \otimes \Sigma^\infty X \simeq \bigoplus_{i \geq 1} \Sigma^i V_i \simeq \prod_{i \geq 1} \Sigma^i V_i.$$

Hence, the loop space Y is given by the product

$$Y \simeq \prod_{i \geq 1} \Omega^\infty \Sigma^i V_i.$$

Moreover, setting for every $n \geq 0$

$$Y_n := \tau_{\leq n} Y = \prod_{i \leq n} \Omega^\infty \Sigma^i V_i,$$

we have $Y \simeq \varprojlim Y_n$. Moreover, since $\tau_{\leq m} k[Y] \simeq \tau_{\leq m} k[Y_m]$ this diagram is pro-constant up to pro-truncation and pointwise bounded below. Thus, [BB24, Lemma 5.10] tells us that its coalgebraic materialization exists and is computed underlying, meaning that

$$k[Y] \simeq M^{\text{cA}}(\varprojlim k[Y_n]) \simeq \varprojlim k[Y_n] \in \text{cCAlg}(\text{Mod}_k)$$

holds, as claimed. $\square$

We now expand from Eilenberg–MacLane spaces to arbitrary connected spaces. Let us recall the following notions in unstable homotopy theory:

Definition 5.5. A space X is called *p-complete* if it is local with respect to the functor $\mathbb{F}_p[-]: \mathcal{S} \rightarrow \text{Mod}_{\mathbb{F}_p}$. Denote by $\mathcal{S}_p \subseteq \mathcal{S}$ the full subcategory category of *p-complete* spaces and denote the reflection functor by $(-)_p^\wedge: \mathcal{S} \rightarrow \mathcal{S}_p$.

Moreover, call a space X *nilpotent* if for each point $x: * \rightarrow X$ the fundamental group $\pi_1(X, x)$ is a nilpotent group which acts nilpotently on the higher homotopy groups of X . Write $\mathcal{S}^{\text{nilp}}$ for the category of nilpotent spaces and $\mathcal{S}_p^{\text{nilp}} = \mathcal{S}_p \cap \mathcal{S}^{\text{nilp}}$ for the category of *p-complete* nilpotent spaces.

Any space X admits a “nilpotent completion” given by taking the limit over all maps from nilpotent spaces into X . Although this in general does not induce an equivalence on homology, the following lemma shows that, if we instead carry out this limit *after* taking homology, the resulting completion map becomes an equivalence.

Lemma 5.6. Let X be a space and R an $\mathbb{E}_\infty$ -ring. There exists a cofiltered diagram of nilpotent spaces $\{X_\alpha\}$, such that the natural map

$$R[X] \rightarrow \varprojlim_{\alpha} R[X_\alpha]$$

is an equivalence in $\text{cCAlg}(\text{Mod}_R)$.

Proof. Since finite limits of nilpotent spaces are nilpotent⁸, [Lur11, Remark 3.1.7] implies that the inclusion $\iota: \mathcal{S}^{\text{nilp}} \rightarrow \mathcal{S}$ induces an adjunction

$$(\hat{}): \text{Pro}(\mathcal{S}) \rightleftarrows \text{Pro}(\mathcal{S}^{\text{nilp}}) : \iota.$$

Here, the left adjoint takes a space X to the formal limit $\hat{X} = \varprojlim_{X_\alpha \rightarrow X} X_\alpha$ running over all maps $X_\alpha \rightarrow X$ where X_α is nilpotent. Moreover, the functor $R[-]: \mathcal{S} \rightarrow \text{Mod}_R$ induces a functor on pro objects given by

$$\hat{R}[-]: \text{Pro}(\mathcal{S}) \rightarrow \text{Pro}(\text{Mod}_R) \quad \varprojlim_i X_i \mapsto \varprojlim_i R[X_i].$$

⁸For example by using [BK72, Corollary 4.5] at each connected component and noting that products of nilpotent groups are nilpotent.

Since the infinite loop space of any spectrum is nilpotent, applying $\widehat{R}[-]$ to the reflection $X \rightarrow \varprojlim_{\alpha} X_{\alpha}$ into $\text{Pro}(\mathcal{S}^{\text{nilp}})$ induces an equivalence

$$cR[X] \xrightarrow{\sim} \widehat{R}[\widehat{X}] = \varprojlim_{\alpha} R[X_{\alpha}]$$

in $\text{Pro}(\text{Mod}_R)$. Since the forgetful functor $c\text{Alg}(\text{Pro}(\text{Mod}_R)) \rightarrow \text{Pro}(\text{Mod}_R)$ commutes with cofiltered limits, the right hand limit can also be computed in $c\text{Alg}(\text{Pro}(\text{Mod}_R))$. Since the coalgebraic materialization of constant objects exists and commutes with limits, we obtain that

$$R[X] \simeq M^{cA}(cR[X]) \simeq M^{cA}(\varprojlim_{\alpha} R[X_{\alpha}]) \simeq \varprojlim_{\alpha} R[X_{\alpha}],$$

as claimed. $\square$

We are now ready to prove the main result of this section.

Theorem 5.7. Let X be a space and k be a field of characteristic p . Then the k -homology $k[X] \in c\text{Alg}(\text{Mod}_k^{\text{cn}})$ is formally étale.

Proof. By Lemma 5.6 we know that we can write

$$k[X] = \varprojlim_{\alpha} k[X_{\alpha}] \in c\text{Alg}(\text{Mod}_k^{\text{cn}})$$

where each X_{α} is a nilpotent. Thus, since formally étale coalgebras are closed under limits, see 4.15, we can assume that X is nilpotent. Moreover, since k is an $\mathbb{F}_p$ -algebra, the natural map $k[X] \rightarrow k[X_p^{\wedge}]$ is an equivalence, and so we can further assume that X is p -complete.

Assume additionally that X is connected and choose a base point. Write $X^{\bullet}$ for the cosimplicial diagram with

$$X^n = (\Omega^{\infty} k \otimes \Sigma^{\infty})^{\circ n} \circ X,$$

obtained via the adjunction $\Sigma^{\infty} \dashv \Omega^{\infty}$. Then, since X is nilpotent and p -complete we have by [BK72, Proposition VI.6.2] that $X \simeq \lim X^n$. Moreover, choosing base points for each X^n , we obtain a splitting of the cosimplicial diagram $k[X^{\bullet}]$, which thus has constant underlying pro-object. In particular, its coalgebraic materialization exists and is computed underlying by [BB24, Lemma 5.10] and we get equivalences of coalgebras $\lim_n k[X^n] \simeq k[\lim_n X^n] \simeq k[X]$. Moreover, by Lemma 5.4 each $k[X^n]$ can be written as a limit $\lim_i k[Y_i^n]$ where each Y_i^n is a finite product of Eilenberg–MacLane spaces. Since the k -homology functor takes finite products of spaces to products of coalgebras and formally étale coalgebras are closed under limits, we reduce to the claim of Proposition 5.3.

Now let S be a non-empty set and suppose we have connected, non-empty spaces X_s for each $s \in S$. Choosing a basepoints for each X_s we can write $X := \coprod_{s \in S} X_s$ as a retract of the constant coproduct $\coprod_S (\prod_{s \in S} X_s)$. Since any functor preserves retracts, we exhibit $k[X]$ as a retract of $k[S] \otimes k[\prod_{s \in S} X_s]$ in $c\text{Alg}(\text{Mod}_k^{\text{cn}})$. The first term is formally étale by 4.21 and the second is the homology of a connected space, so we conclude. $\square$

Thus, we have answered our initial question about recovering spherical chains from characteristic p -chains, which we may summarize as follows.

Corollary 5.8. Let $X \in \mathcal{S}$ be a space and k be a perfect field of characteristic p . The space of lifts of $k[X] \in c\text{Alg}(\text{Mod}_k^{\text{cn}})$ to a coalgebra in p -complete, connective $\mathbb{W}(k)$ -modules is contractible and the unique point is given by $\mathbb{W}(k)[X]^{\wedge}$. Moreover, for any $A \in c\text{Alg}(\text{Mod}_{\mathbb{W}(k)}^{\wedge, \text{cn}})$ the base change along the map $\mathbb{W}(k) \rightarrow k$ induces an equivalence

$$\text{Map}_{c\text{Alg}(\text{Mod}_{\mathbb{W}(k)}^{\wedge})}(A, \mathbb{W}(k)[X]^{\wedge}) \xrightarrow{\sim} \text{Map}_{c\text{Alg}(\text{Mod}_k)}(A \otimes k, k[X]).$$

Proof. Combine Theorem 5.7 with Theorem 5.1. $\square$

Finally, we observe that this lifts the result of [BB24] that, for a separably closed field k , taking k -chains is fully faithful as functor to k -coalgebras when restricted to p -complete nilpotent spaces, which was in turn a coalgebraic generalization of Mandell's result [Man01], to the spherical Witt-vectors of k .

Corollary 5.9. Let k be a perfect, separably closed field of characteristic p with spherical Witt vectors $\mathbb{W}(k)$. The p -complete $\mathbb{W}(k)$ -homology functor

$$\mathcal{S}_p^{\text{nilp}} \rightarrow \text{cCAlg}(\text{Mod}_{\mathbb{W}(k)}^\wedge) \quad X \mapsto \mathbb{W}(k)[X]_p^\wedge$$

is fully faithful. In particular, for any nilpotent space X , we have a natural equivalence

$$X_p^\wedge \simeq \text{Map}_{\text{cCAlg}(\text{Mod}_{\mathbb{W}(k)}^\wedge)}(\mathbb{W}(k), \mathbb{W}(k)[X]_p^\wedge).$$

Proof. By [BB24, Theorem 1.2] the functor

$$k[-]: \mathcal{S}_p^{\text{nilp}} \rightarrow \text{cCAlg}(\text{Mod}_k)$$

is fully faithful. By Theorem 5.7 it factors through the full subcategory $\text{cCAlg}(\text{Mod}_k^{\text{cn}})^{\text{fét}}$ and hence we may compose with the functor $\mathcal{R}_{\mathbb{W}(k)/k}$ of Theorem 5.1 to get the claim. $\square$

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