GALOIS CATEGORIES AND THE ÉTALE FUNDAMENTAL GROUP

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Contents

1. Introduction

1.1. **Overview.** At the centre of modern algebraic geometry lies the contravariant equivalence between algebras and geometric spaces. One of the most intricate and well understood theories on the algebraic side is Galois theory, i.e. the study of suitably nice field extensions. This theory bears many formal resemblances with the seemingly unrelated theory of covering spaces in algebraic topology, except all arrows appear to be reversed. The connection is that if we think of a field k as the set of functions on the geometric space $\mathrm{Spec}(k)$, then we can view Galois theory instead as the study of covering spaces of spectra of fields. This point of view begs the question what a covering space of a general scheme is and whether it is possible to classify them in the same way as in algebraic topology. This story is usually called *Grothendiecks Galois Theory* and the goal of this thesis is to give an introduction to this chain of ideas.
Naturally there are some obstacles towards simply importing ideas from topology. First of all, the purely topological notion of a covering space is clearly not sufficient even for ordinary Galois theory, as the spectrum of a field is a point, which should ultimately be understood as a failure of the Zariski topology to encode enough geometric structure. Furthermore, since fields are very special rings there is no obvious way to define a Galois extension of a ring and hence a covering of an affine scheme. A hint from topology is that for some scheme X the projection $X \sqcup \cdots \sqcup X \to X$ should be a covering and we might furthermore ask that any covering 'locally' look like this. This is exactly the right idea if we know what we mean by

'locally' (It is not correct in terms of the Zariski topology). More to the point, the correct notion will be that of an *étale covering* of a scheme X which in the case $X = \mathrm{Spec}(k)$ corresponds to a finite product of finite separable field extensions. One can then construct the so called *étale fundamental group* $\pi_1(X)$ which agrees with the absolute Galois group in the case $X = \mathrm{Spec}(k)$ and naturally classifies the étale coverings of X. Having once fixed the proper notion of covering this is done in purely categorical terms by means of so called *Galois categories*, which we introduce in the next section. Thus the machinery developed here can be applied to other settings as well.
To start off let us recall the fundamental theorems of covering spaces and Galois theory respectively:

(1) Let X be a locally path-connected and semi-locally-simply-connected space and denote by Cov_X the category of covering spaces of X. Then consider the groupoid $\Pi_1(X)$ called the *Fundamental Groupoid* of X with objects given by the points of X and maps given by homotopy types of paths. The main theorem says that there is an equivalence of categories :

$$\mathrm{Cov}_X \xrightarrow{\sim} \mathrm{Hom}(\Pi_1(X), \mathrm{Set}) \qquad (Y \xrightarrow{p} X) \;\mapsto\; \Pi_1(X) \curvearrowright \coprod_{x\in X} p^{-1}(x) \tag{1}$$

Which is induced by the action of homotopy types of paths on fibres via the homotopy lifting property. This functor is represented by a simply connected space $\hat{X} \to X$ called the *universal covering* of X. Furthermore if (X, x) is a pointed space then we can understand x as an object of $\Pi_1(X)$ and $\pi_1(X, x) = \mathrm{Aut}_{\Pi_1(X)}(x)$ is called the *fundamental group* of the pair (X, x). In particular we denote by $\mathrm{Cov}_{(X,x)}$ the category of covering spaces of (X,x) then (1) reduces to the following equivalence:

$$\mathrm{Cov}_{(X,x)} \xrightarrow{\sim} \mathrm{Set}^{\pi_1(X,x)} \qquad (Y \xrightarrow{p} X) \;\mapsto\; \pi_1(X, x) \curvearrowright p^{-1}(x) \tag{2}$$

(2) Let k be a field and let $\bar{k}$ be a choice of algebraic closure of k and k^s be the separable closure of k in $\bar{k}$. A finite étale k-algebra is defined as a finite product $L_1 \times \cdots \times L_n$ of finite separable field extensions L_i/k and we let $\mathcal{F}\text{ét}\, k\, \mathrm{Alg}$ denote the category of such algebras. The *absolute Galois group* is defined as the group $\mathrm{Gal}(k^s/k) := \mathrm{Aut}_k(L)$. This group is isomorphic to the cofiltered limit over all finite Galois extensions and thus a profinite group with a natural topology. Then for any $A \in \mathcal{F}\text{ét}\, k\, \mathrm{Alg}$ there is a natural continuous right-action of $\mathrm{Gal}(k^s/k)$ on the embeddings of the components of A into k^s. This induces an equivalence of categories:

$$\mathcal{F}\text{ét}\, k\, \mathrm{Alg}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{E}\mathrm{ns}_{\mathrm{fin}}^{\mathrm{Gal}(k^s/k)} \qquad A \;\mapsto\; \mathrm{Gal}(k^s/k) \curvearrowright \mathrm{Hom}(A, k^s) \tag{3}$$

The content of the equivalence (3) is usually stated somewhat differently by saying that finite separable filed extensions of k are in inclusion reversing one-to-one correspondence with open subgroups of $G = \mathrm{Gal}(k^s/k)$. Since the continuous, finite and transitive G-sets are precisely those isomorphic to G/H where $H \le G$ is open, i.e. of finite index, this is a special case of our original statement. The two statements given above are quite similar but not entirely symmetric.
First of all there are more finiteness conditions involved with the absolute Galois group. This is simply due to the algebraic nature of the theory. Furthermore the definition of the topological fundamental group very explicitly involves the choice of a base point and the fundamental groupoid is then the groupoid formed over all such choices. The fact that these choices do not matter in terms of the isomorphism type is reflected by the fact that this groupoid is connected. However, it is not all obvious what this corresponds to in Galois theory. As it

turns out the relevant choice here is picking an algebraic closure $\bar{k}/k$ and we will discuss this in detail once we have developed the general theory. Lastly the reason that the functor $\mathcal{F}\text{ét}\,k\,\text{Alg} \to \mathcal{E}\text{ns}_{\text{fin}}^{\text{Gal}(k^s/k)}$ is contravariant is of course that $\mathcal{F}\text{ét}\,k\,\text{Alg}$ is not the category of spaces, but that of functions on our spaces.
How should this all generalize to schemes then? The topological fundamental group is quite approachable as an invariant since it is both easily defined as the space of homotopy types of pointed maps $S^1 \to X$ and also has a very nice geometric interpretation. However uncovering its true structural meaning, i.e. the fact that it classifies the covering spaces of X, takes some work. The category of schemes however is completely different from the category of topological spaces and thus the 'flavor' of geometry is different as well. In particular higher categorical notions such as homotopy are not naturally available and require a very different approach. The idea will be then to work backwards: We know that the fundamental group is supposed to classify the coverings of our space in the precise sense that the category of coverings is equivalent to that of actions of the fundamental group. Thus we will think abstractly about categories which behave like these two and try to recover a group from their structural properties, so that fundamental groups will classify coverings by definition and we will have to deliver some geometric intuition afterwards.

1.2. **Conventions.**

- Any finite set is understood to be equipped with the discrete topology if it is considered as a topological space.
- All groups act from the left unless otherwise specified.
- For a topological group we denote by $\text{Set}_{\text{fin}}^G$ the category of finite G-sets and by $\mathcal{E}\text{ns}_{\text{fin}}^G$ the category of finite *continuous* G-sets as above. Consequently we have the following notations for the category of finite sets: $\text{Set}_{\text{fin}} = \text{Set}_{\text{fin}}^1 = \mathcal{E}\text{ns}_{\text{fin}}^1 = \mathcal{E}\text{ns}_{\text{fin}}$.
- The words *map*, *arrow* and *morphism* will be used interchangeably whenever there is no risk of confusion. We refer to arrows in the category Set as *set-maps* for clarity.
- If $\mathcal{C}$ is a category and X is an object of $\mathcal{C}$ for the sake of brevity we write $X \in \mathcal{C}$ instead of $X \in \text{ob}(\mathcal{C})$.
- For two functors $F, G : \mathcal{C} \to \mathcal{D}$ we denote by $[F, G] = \text{Nat}(F, G)$ the set of natural transformations $F \to G$.
- We denote filtered colimits by $\varinjlim$ and cofiltered limits by $\varprojlim$.
- For a finite set S we denote its cardinality as $\#S$.
- For some group G we denote its profinite completion as $\hat{G}$
- If $\mathcal{C}$ is a category we denote the terminal object of $\mathcal{C}$ by $*_{\mathcal{C}}$ and the initial object by $\varnothing_{\mathcal{C}}$. If there is no risk of confusion we just write $*$ and $\varnothing$ respectively.

2. Galois categories

2.1. **Preliminaries.** Let G be a group. The fundamental observation is the following.

Proposition 2.1. *Consider the forgetful functor $L : \text{Set}^G \to \text{Set}$. If we let G act on itself via left-multiplication then L is represented by $G \in \text{Set}^G$ and furthermore $\text{Aut}(L) \cong G$.*

Proof. Indeed for any G-Set S an equivariant map $\varphi : G \to S$ is uniquely determined by $\varphi(1) \in S$ so we have $\text{Hom}(G, S) \cong S$ naturally in S. Furthermore by the Yoneda lemma $[\text{Hom}(G, -), \text{Hom}(G, -)] \cong \text{Hom}(G, G)$ and these maps are precisely those given by right-multiplication with some element $g \in G$, but any such map is a bijection. □

Now consider a topological space X that is path-connected, locally path-connected, and semi-locally simply connected. For any base point $x \in X$ the so called *fibre functor* with respect to x is the functor $F_x : \mathrm{Cov}_{(X,x)} \to \mathrm{Set}$ sending a covering $Y \xrightarrow{p} X$ to the fibre $p^{-1}(x)$. Then our equivalence (2) gives a commutative diagram:

$$\begin{array}{ccc} \mathrm{Cov}_{(X,x)} & \xrightarrow{\sim} & \mathrm{Set}^{\pi_1(X,x)} \\ & \overset{F_x}{\searrow} & \downarrow L \\ & & \mathrm{Set} \end{array} \tag{4}$$

Furthermore the functor F_x is represented by the universal covering $\hat{X} \to X$. It follows that $\mathrm{Aut}(F_x) \cong \mathrm{Aut}(L) \cong \pi_1(X,x)$. The idea is then to single out formal properties of the pair $(\mathrm{Cov}_{(X,x)}, F_x)$ which suffice to induce an equivalence as such. This is however not quite the correct notion for our purposes. The issue is that covering maps in scheme theory are necessarily finite, while the universal coverings tend to be infinite. As an informal example consider the (topological) coverings:

$$\mathbb{C}^\times \to \mathbb{C}^\times \quad x \mapsto x^n \quad ; \quad \mathbb{C} \to \mathbb{C}^\times \quad x \mapsto \exp(x)$$

Now the former corresponds to an étale endomorphism of $\mathrm{Spec}(\mathbb{C}[T]) \setminus 0 = \mathbb{A}^1_{\mathbb{C}} \setminus 0$ via the map $T \mapsto T^n$ while the latter has no algebraic analogue. The correct objects of study are then, for some profinite group G, the category $\mathcal{E}\mathrm{ns}^G_{\mathrm{fin}}$ of finite G-sets on which G acts continuously along with the forgetful functor to $\mathcal{E}\mathrm{ns}_{\mathrm{fin}}$. This functor is then only prorepresentable so the situation becomes more subtle. First we need some definitions:

Definition 2.2. Let $\mathcal{C}$ be a category with fibre products, then a morphism $f : B \to C$ in $\mathcal{C}$ is called a *strict epimorphism* if the following diagram is a pushout:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow f & & \downarrow \mathrm{id} \\ Y & \xrightarrow{\mathrm{id}} & Y \end{array}$$

The notion of a *strict* monomorphism is precisely the dual one.

This is a classic fix for the problem that epimorphisms tend to not behave as nicely as surjective maps in the category of sets. Furthermore since the definition can be stated in terms of a colimit it will be easier to understand how certain functors interact with strict epimorphisms, which is the sort of thing we will be interested in. For example it is clear that cocontinuous functors map strict epimorphisms to strict epimorphisms and it is also easy to see that strict epimorphism + monomorphism implies isomorphism. Furthermore any coequalizer is easily seen to be an epimorphism, so this is indeed a stronger property. We note that there is some ambiguous terminology in use regarding the terms *strict* and *effective*. We will only use the term *strict epimorphism* exactly as defined above.

Definition 2.3. Let $\mathcal{C}$ be a category and $X, Y \in \mathcal{C}$ objects. Furthermore let G be a group and write BG for the associated groupoid with one element $* \in BG$. Then we say that a map $X \to Y$ exhibits Y as the quotient of X by a G-action if it is the colimit of some diagram $BG \to \mathcal{C}$ mapping $* \mapsto X$. In this case and if the action is implicit we also write $Y = X/G$.

Remark 2.4. If $\mathcal{C}$ is a concrete category such that the forgetful functor $C \to \mathrm{Set}$ commutes with colimits then the above clearly agrees with the usual notion of a quotient. In particular for a set X with a G-action $\rho : G \to \mathrm{Aut}(X)$ the quotient X/G as defined above is given by the sets of equivalence classes of points identified by ρ, as one would expect.

2.2. **The axioms.** We are now ready to state the axioms for a Galois category. Since we are trying to construct categories equivalent to $\mathcal{E}\mathrm{ns}^G_{\mathrm{fin}}$, the obvious thing to do is to copy all the categorical properties. However some of these imply each other in not so obvious ways so we have a choice here in how optimal we want to make these axioms. We shall stick to the treatment in [1] and try to give a minimal set of axioms, making them easier to verify in practice but the theoretical work somewhat harder.

Definition 2.5. Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathcal{E}\mathrm{ns}_{\mathrm{fin}}$ a functor. Then the pair $(\mathcal{C}, F)$ is called a *Galois category* if it satisfies the following axioms:

(i) $\mathcal{C}$ has a terminal object and all fibre products (Equivalently $\mathcal{C}$ has all finite limits).
(ii) $\mathcal{C}$ has all finite coproducts and all quotients by finite groups.
(iii) Any arrow $X \xrightarrow{f} Y$ in $\mathcal{C}$ factors as $X \xrightarrow{u} Z \xrightarrow{v} Y$ where u is a strict epimorphism and v is a monomorphism which is an isomorphism onto a direct summand of Y, i.e. there exists another monomorphism $Z' \to Y$ such that the induced map $Z \sqcup Z' \to Y$ is an isomorphism.
(iv) F commutes with finite limits.
(v) F commutes with finite coproducts and quotients by finite groups. Furthermore F maps strict epimorphisms to surjective maps.
(vi) F reflects isomorphisms.

The functor F is referred to as a *fibre functor*. Note that (i)-(iii) describe the structure of $\mathcal{C}$ and (iv)-(vi) describe how F interacts with this structure. In a slight abuse of terminology we also call $\mathcal{C}$ a Galois category if it satisfies (i)-(iii) and admits some functor $F : \mathcal{C} \to \mathcal{E}\mathrm{ns}_{\mathrm{fin}}$ satisfying (iv)-(vi). This is because the choice of a fibre functor amounts to the choice of a base point in the geometric context, and we want to stress the fact that this is not canonical. To make sense of these axioms we should first of all check that for some profinite group G our primordial category $\mathcal{E}\mathrm{ns}^G_{\mathrm{fin}}$ together with the forgetful functor $F : \mathcal{E}\mathrm{ns}^G_{\mathrm{fin}} \to \mathcal{E}\mathrm{ns}_{\mathrm{fin}}$ satisfies them.

Proposition 2.6. *Let G be any group and Set^G be the category of G-sets. Then Set^G is bicomplete. It follows that $\mathrm{Set}^G_{\mathrm{fin}}$ has all finite (co-)limits.*

Proof. Indeed Set^G is equivalent to the functor category $\mathrm{Hom}(\mathcal{G}, \mathrm{Set})$ which is well known to have all limits and colimits. In particular these are computed point wise, i.e. by taking the corresponding limit in Set. The second statement is clear. □

The case of continuous actions is just as nice:

Proposition 2.7. *Let G be a topological group, then the forgetful functor*

$$\mathcal{E}ns^G_{\mathrm{fin}} \to \mathrm{Set}^G_{\mathrm{fin}} \to \mathrm{Set}$$

creates all finite limits and colimits and, in particular, commutes with them. Furthermore this functor reflects isomorphisms.

Proof. The first part amounts to finding the correct topologies to equip our G-sets with. Consider for example finite limits: It suffices to construct equalizers and finite products. These naturally carry a subspace and product topology respectively and it is easy to see that the corresponding actions are then continuous. The second statement just says that equivariant map has an equivariant inverse if and only if it is a bijection, which is clear. □

Corollary 2.8. *A map in $\mathcal{E}ns^G_{\mathrm{fin}}$ is a monomorphism (resp. epimorphism) if and only the corresponding set-map is injective (resp. surjective).*

Proof. A set-map $f : X \to Y$ is a monomorphism if and only if the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback and dually for epimorphisms Thus the claim follows immediately from our proposition. □

This proves that $\mathcal{E}\mathrm{ns}^G_{\mathrm{fin}}$ along with the forgetful functors satisfies the axioms (i),(ii),(iv), (v) and (vi). For (v) note that any arrow of G-sets $f : S \to S'$ factors into set-maps $S \twoheadrightarrow \mathrm{im}(f) \hookrightarrow S'$. Now since f is equivariant $\mathrm{im}(f)$ naturally inherits a G-set structure from S' making this a diagram in $\mathcal{E}\mathrm{ns}^G_{\mathrm{fin}}$. Furthermore $S \setminus \mathrm{im}(f)$ also gets a natural G-action such that the inclusion is equivariant, i.e. we have $S' \cong \mathrm{im}(f) \sqcup (S' \setminus \mathrm{im}(f))$ in $\mathcal{E}\mathrm{ns}^G_{\mathrm{fin}}$. The fact that $S \twoheadrightarrow \mathrm{im}(f)$ is a strict epimorphism follows immediately from Proposition 2.7 since in Set all epimorphisms are strict.

2.3. **The main theorem.** In the following let $(\mathcal{C}, F)$ denote a Galois category. Our first goal is to show that $\mathrm{Aut}(F)$ is a profinite group which acts continuously on the sets $F(X)$ for $X \in \mathcal{C}$. First some observations:

Definition 2.9. Let $\mathcal{D}$ be any category. Then some object $X \in \mathcal{D}$ is called *artinian* if any chain of monomorphisms :

$$\cdots \hookrightarrow X_i \hookrightarrow X_{i-1} \hookrightarrow \cdots \hookrightarrow X_0 = X$$

stabilizes eventually. The category $\mathcal{D}$ is called artinian if every object of $\mathcal{D}$ is artinian.

Proposition 2.10. *F reflects monomorphisms.*

Proof. This is the same argument as in Proposition 2.8 using axioms (iv) and (vi). □

Corollary 2.11. *Galois categories are artinian.*

Proof. Follows immediately from the previous proposition and the fact that $\mathcal{E}\mathrm{ns}_{\mathrm{fin}}$ is clearly artinian. □

We want to think of our category as consisting of coverings of some geometric object. From topology we know that the non-connected coverings of a space are in a sense 'redundant', for they are easily constructed from the connected ones. We will now assert this notion in our abstract setting.

Definition 2.12. $X \in \mathcal{C}$ is called *connected* if X is nonempty and $X \cong X_1 \sqcup X_2$ implies $X_i = \varnothing$ for some $i \in \{1, 2\}$.

Remark 2.13. It is easy to see that in an artinian category any objects admits an essentially unique decomposition into connected objects. Furthermore for a group G the connected G-sets are precisely the transitive ones. The decomposition is then that into the orbits of the G-action.

By the (contravariant) Yoneda embedding $X \mapsto \mathrm{Hom}(X, -)$ we may think of any functor $\mathcal{C} \to \mathcal{E}\mathrm{ns}_{\mathrm{fin}}$ as a 'generalized object' of $\mathcal{C}$ and from this perspective we have already seen that the fibre functor F should correspond to the universal covering, i.e. to torsors of the fundamental group. Generally this has no chance of being an actual object of $\mathcal{C}$. However we will now see that F is always a strict pro-object of C and deduce some facts about the defining projective system.

Theorem 2.14. *The functor $F : \mathcal{C} \to \mathcal{E}ns_{\text{fin}}$ is strictly prorepresentable by connected objects, i.e. there exists a cofiltered diagram $P : I \to \mathcal{C}$ such that $F \cong \varinjlim \operatorname{Hom}(P, -)$, each P_i is connected and the transition maps $P_i \to P_j$ are strict epimorphisms. In particular we regard F as the limit of the objects P_i.*

Proof. Consider the Grothendieck construction $\operatorname{el}(F)$ i.e. the category whose objects are pairs (X, x) where $X \in \mathcal{C}$ and $x \in F(X)$ and whose morphisms $(X, x) \to (Y, y)$ are morphisms $f : X \to Y$ such that $Ff(x) = y$, together with the projection $\operatorname{el}(F) \to \mathcal{C}$. Then the composition with the Yoneda embedding

$$\operatorname{el}(F)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \to \operatorname{Hom}(\mathcal{C}, \mathrm{Set})$$

gives a diagram of representable functors. It is a well known fact that F is indeed the colimit of this diagram. Thus we want to show that the connected objects are cofinal i.e. for each $(X, x) \in \operatorname{el}(F)$ there exists some $Z \in \mathcal{C}$ connected and a map $(Z, z) \to (X.x)$ in $\operatorname{el}(F)$. To see this consider some object X and consider the decomposition $X \cong X_1 \sqcup \cdots \sqcup X_n$ into connected objects mentioned in Remark 2.13. In particular since F commutes with coproducts we also have $F(X) \cong F(X_1) \sqcup \cdots \sqcup F(X_n)$. Now given some element $x \in F(X)$ we have that $x \in F(X_i)$ for some $1 \leq i \leq n$. It follows that the inclusion $X_i \hookrightarrow X$ defines a morphism to (X, x) in $\operatorname{el}(F)$ as desired. It remains to show that any map $Y \to X$ in $\mathcal{C}$ is a strict epimorphism if X is connected, but this follows immediately from the decomposition of maps in axiom (iii) since the second map is necessarily an isomorphism (because X is connected). □

Geometrically this tells us that our universal covering is in a sense approximated well by the coverings P_i. On the more technical side the projective system P and the properties just proven will be the key to our main theorem.

Lemma 2.15. *Let $X \in \mathcal{C}$ be connected. Then any map $X \to X$ is an isomorphism i.e. $\operatorname{Hom}(X, X) = \operatorname{Aut}(X)$.*

Proof. Consider a map $X \to X$, we have already seen that any such map must be a strict epimorphism. Then by axiom (v) the corresponding set-map $F(X) \to F(X)$ is surjective and thus bijective since the sets are finite. But F reflects isomorphisms by axiom (vi) so we are done. □

Our current goal is to understand how the group $\operatorname{Aut}(F)$ acts on the fibres $F(X)$. Since F is prorepresented by a system of connected objects it is natural to first consider the actions of connected objects. In particular we have the following observation:

Proposition 2.16. *Let $f, g : X \to Y$ be maps in $\mathcal{C}$ and X be connected. Then if $Ff(x) = Fg(x)$ for any $x \in F(X)$ it follows that $f = g$.*

Proof. Let E be the equalizer of f and g. Then since F commutes with finite limits $F(E) \subseteq F(X)$ is the set of points where Ff and Fg agree. By assumption $F(E)$ is non-empty and it follows easily from our axioms that then E is nonempty as well. But then since X is connected the canonical monomorphism $E \to X$ is an isomorphism and the claim follows. □

If we take $Y = X$ this says precisely that the natural action of $\operatorname{Hom}(X, X) = \operatorname{Aut}(X)$ on $F(X)$ sending $f : X \to X$ to $Ff : F(X) \to F(X)$ induced by F is always free. In particular we have the inequality $\# \operatorname{Aut}(X) \leq \# F(X)$. The case where the action is transitive i.e. equality holds is important enough to warrant a definition:

Definition 2.17. Let $X \in \mathcal{C}$ be connected, then X is called *Galois* if the action of $\operatorname{Aut}(X)$ on $F(X)$ is transitive.

Remark 2.18. We note some immediate consequences from the definition:

(1) By Proposition 2.16 the following are equivalent:
  (a) X is Galois
  (b) The action $\mathrm{Aut}(X) \curvearrowright F(X)$ is regular
  (c) $\# \mathrm{Aut}(X) = \# F(X)$
  (d) $X / \mathrm{Aut}(X) = *$

  Where the last statement follows from axiom (v) i.e. the fact that F commutes with quotients by finite groups.

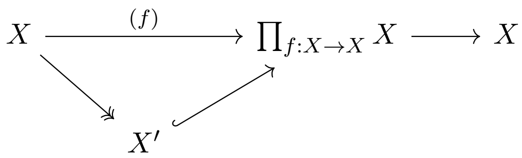
(2) By the statement (d) above the notion of a Galois object does not depend on the choice of a fibre functor F. Later we will further investigate the consequences of such a choice.

(3) We will later see that Galois objects correspond to Galois extensions in the algebraic and regular covers in the topological setting.

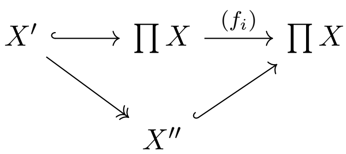
The following purely categorical theorem has a distinctly algebraic flavor. We shall explore this later on when working directly with schemes.

Theorem 2.19. *For any connected $X \in \mathcal{C}$ and a point $x \in F(X)$ there exists a Galois object $Y \in \mathcal{C}$, a point $y \in F(Y)$ and a map $f : Y \to X$ such that $Ff(y) = x$.*

Proof. We have already seen that any map into a connected object is a strict epimorphism. Thus since F maps strict epimorphisms to surjective maps it suffices to show that there exists some map $f : Y \to X$. Consider then the factorization of the natural map:



We claim that X' is our desired Galois object. To show that $\mathrm{Aut}(X')$ acts transitively on the fibre $F(X')$ consider two points $(x_i), (y_i) \in F(X') \subseteq \prod F(X)$. Then since X is connected we can find maps $f_i : X \to X$ such that $Ff_i(x_i) = y_i$. Consider again the factorization:



To see that $(f)_i$ defines automorphism of X' it suffices to show that $X' \to X''$ is an isomorphism. However $F(X') \to F(X'')$ is clearly a bijection and thus since F reflects isomorphisms we are done. $\square$

Corollary 2.20. *Let P be the diagram of F as in Theorem 2.14. Then the system of P_i which are Galois is cofinal in P i.e. we may assume that all P_i are Galois.*

Now any isomorphism $P_i \to P_i$ naturally defines a cone over P i.e. there is a canonical injective map $\mathrm{Hom}(P_i, P_i) = \mathrm{Aut}(P_i) \to F(P_i)$ and since the P_i are Galois this is actually a bijection. Using this fact we get:

$$\mathrm{Hom}(F, F) = \varprojlim_i \varinjlim_j \mathrm{Hom}(P_j, P_i) = \varprojlim_i F(P_i) = \varprojlim_i \mathrm{Aut}(P_i) \tag{5}$$

Thus $\mathrm{Hom}(F, F) = \mathrm{Aut}(F)$ is in fact a profinite group and we endow it with the corresponding topology. This is indeed the group we have been looking for.

Definition 2.21. The profinite group $\mathrm{Aut}(F)$ is called the *fundamental group* of $\mathcal{C}$ with respect to the fibre functor F. We denote this group by $\pi_1(\mathcal{C}, F)$.

Of course we could have made this definition much earlier. However we now see the following: Because of the natural maps:

$$\mathrm{Aut}(F) \to \prod_{X\in\mathcal{C}} \mathrm{Aut}(F(X)) \xrightarrow{\mathrm{pr}} \mathrm{Aut}(F(X))$$

We see that the fundamental group acts on the fibres $F(X)$. Then since trivially (everything is finite i.e. discrete) each $\mathrm{Aut}(P_i)$ acts continuously on $\mathrm{Hom}(P_i, X)$, we get that the action of $\pi_1(\mathcal{C}, F)$ is in fact continuous. Let $\pi = \pi_1(\mathcal{C}, F)$, then this defines a functor:

$$\tilde{F} : \mathcal{C} \to \mathcal{E}\mathrm{ns}^{\pi}_{\mathrm{fin}} \qquad X \mapsto \pi \curvearrowright F(X) \tag{6}$$

The main result about Galois categories is the following:

Theorem 2.22. *The functor $\tilde{F}$ defined in* (6) *is an equivalence of categories. Furthermore F agrees with the standard fibre on $\mathcal{E}ns^{\pi}_{\mathrm{fin}}$ i.e. we get a commutative diagram:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\tilde{F}} & \mathcal{E}ns^{\pi}_{\mathrm{fin}} \\ & {\scriptstyle F}\searrow & \downarrow \\ & & \mathcal{E}ns_{\mathrm{fin}} \end{array} \tag{7}$$

Where $\mathcal{E}ns^{\pi}_{\mathrm{fin}} \to \mathcal{E}ns_{\mathrm{fin}}$ is the forgetful functor.

Proof. Commutativity of the diagram is clear from the definitions. Note that, since $\mathcal{E}\mathrm{ns}^{\pi}_{\mathrm{fin}}$ along with the forgetful functor is a Galois category, $\tilde{F}$ also satisfies the axioms (iv)-(vi), which we shall use freely. We have also seen that the forgetful functor creates all finite (co)-limits so we shall not distinguish between $F(X)$ and $\tilde{F}(X)$ when calculating them. We show explicitly that $\tilde{F}$ is fully faithful and essentially surjective:

(1) *faithful*: By the factorization just given it suffices to prove that F is faithful. This is clear since F commutes with equalizers and reflects isomorphisms.
(2) *full*: Let $X, Y \in \mathcal{C}$ and $s : F(X) \to F(Y)$ be a π-invariant map. Consider the graph $\Gamma_s = F(Y) \times_{F(Y)} F(X) \hookrightarrow F(Y) \times F(X)$. Then Γ is a union of connected components of $F(Y) \times F(X)$. Since F commutes with finite products and maps connected objects to transitive π-sets it follows from remark 2.13 that F respects the decomposition into connected objects. Thus there exists a subobject $Z \hookrightarrow X \times Y$ which is a union of connected components such that $F(Z) = \Gamma_s$. Then since $F(Z) \xrightarrow{F} (X)$ is a bijection and F reflects isomorphisms the map

$$Z \hookrightarrow Y \times X \to X$$

is an isomorphism. Hence if we let f be the composition:

$$X \xrightarrow{\sim} Z \hookrightarrow Y \times X \to Y$$

we get $Ff = s$ as desired. Thus $\hat{F}$ is full.
(3) *essentially surjective*: Again, since the connected objects of $\mathcal{E}\mathrm{ns}^{\pi}_{\mathrm{fin}}$ are precisely the transitive π-sets and $\tilde{F}$ preserves the decomposition, it suffices to prove that the image of $\tilde{F}$ contains all transitive sets. Since π is profinite these are precisely those of the form π/τ where $\tau \subseteq \pi$ is an open subgroup and we only need to show that π/τ lies in the essential image of F. If F were representable by an object $P \in \mathcal{C}$ and we knew that F commutes with quotients by *arbitrary* groups we would have

$F(P/\tau) = F(P)/\tau = \pi/\tau$ and that would be the end of it. Consider then the system P_i pro-representing F and the natural action of τ on this system. Since each group $\mathrm{Aut}(P_i)$ is finite this action factors through some finite group τ_i for each i. In particular the quotients P_i/τ_i exist and, since $\tau \subseteq \pi$ is an open subgroup, it follows from the definition of the product topology that the corresponding diagram in $\mathcal{C}$ stabilizes, i.e. the limit $\varprojlim P_i/\tau_i$ exists in $\mathcal{C}$. Now since F commutes with quotients by finite groups and finite limits and cofiltered limits commute with finite colimits we get:

$$F(\varprojlim_i P_i/\tau_i) = \varprojlim_i F(P_i/\tau_i) = \varprojlim_i F(P_i)/\tau_i = \varprojlim_i F(P_i)/\tau = (\varprojlim_i (F(P_i))/\tau = \pi/\tau$$

As desired and thus $\hat{F}$ is essentially surjective.

□

Corollary 2.23. *Let $(\mathcal{C}, F)$ be a Galois category. Then F creates all finite limits and colimits. In particular any category $\mathcal{C}$ that admits a fibre functor is finitely bicomplete.*

Corollary 2.24. *Let G be any group and consider the category $\mathcal{C} = \mathrm{Set}_{\mathrm{fin}}^{G}$ of all actions of G on finite sets. Then $\mathcal{C}$ is equivalent to $\mathcal{E}ns_{\mathrm{fin}}^{\hat{G}}$ where $\hat{G}$ is the profinite completion of G and we only consider continuous actions.*

There is still one unpleasantness we need to deal with. In the real world we will generally encounter only a category $\mathcal{C}$ on its own and will need to construct a fibre functor for $\mathcal{C}$ to apply the machinery developed above. Furthermore we would like to think of $\pi_1(\mathcal{C}, F)$ as an invariant of $\mathcal{C}$ depending only 'tamely' on the choice of F as the following proposition shows:

Proposition 2.25. *Let $\mathcal{C}$ be a Galois category and F, F' be two fibre functors for $\mathcal{C}$. Then there exists an isomorphism $F \xrightarrow{\sim} F'$.*

Proof. In Remark 2.18 we already saw the notion of a Galois Object in $\mathcal{C}$ does not depend on the choice of fibre functor. The claim then follows easily from the fact both functors are represented by the Galois Objects of $\mathcal{C}$. □

We can say more: The following claim tells us that the fundamental group is *functorial* in a precise sense.

Proposition 2.26. *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories and let $G : \mathcal{C} \to \mathcal{C}'$ be a functor such that we have a commutative diagram:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{C}' \\ \downarrow F & \swarrow F' & \\ \mathcal{E}ns_{\mathrm{fin}} & & \end{array}$$

Then we get a continuous group homomorphism $\pi_1(\mathcal{C}', F') \to \pi_1(\mathcal{C}, F)$ and an induced commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad G \quad} & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathcal{E}ns_{\mathrm{fin}}^{\pi_1(\mathcal{C},F)} & \longrightarrow & \mathcal{E}ns_{\mathrm{fin}}^{\pi_1(\mathcal{C}',F')} \end{array}$$

Proof. Given the datum $(\eta'_X \in \mathrm{Aut}(F(X)))_{X \in \mathcal{C}'}$ of an automorphism $\eta' : F' \to F'$ we get an induced automorphism $\eta : F \to F$ by setting $\eta_Y := \eta'_{G(Y)}$ for $Y \in \mathcal{C}$. This defines a continuous group homomorphism $\mathrm{Aut}(F') \to \mathrm{Aut}(F)$ which by pullback induces a functor $\mathcal{E}\mathrm{ns}_{\mathrm{fin}}^{\pi_1(\mathcal{C},F)} \to \mathcal{E}\mathrm{ns}_{\mathrm{fin}}^{\pi_1(\mathcal{C}',F')}$ as desired. Commutativity of the diagram is clear from the definitions. □

Thus if $(\mathcal{C}, F)$ is a Galois Category we can think of $\pi_1(\mathcal{C}, F)$ as an invariant of the category $\mathcal{C}$ whose isomorphism type does not depend on F. Thus when we are trying to calculate π_1 we can choose fibre functors at our convenience. However for theoretical purposes it is nice to have an invariant which is itself independent of any choices, not just its isomorphism type. The sophisticated solution is the following:

Definition 2.27. Let $\mathcal{C}$ be Galois category. We define the *fundamental groupoid* $\Pi_1(\mathcal{C})$ as the category of fibre functors on $\mathcal{C}$. Then $\Pi_1(\mathcal{C})$ is a connected groupoid and in particular the fundamental group is well defined up to inner automorphism.

3. The étale fundamental group

3.1. **Étale coverings.** We now want to make use of the machinery of Galois categories to define the fundamental group of an arbitrary (connected locally noetherian) scheme. To see how to do this consider again our topological inspiration: For a connected, locally path-connected and semi-locally-simply-connected space X we see from the equivalence (2) that for any $x \in X$ the category of finite coverings of X defines a Galois category with fibre functor given by taking the actual fibre over x. The fundamental group recovered in this way is then the profinite completion of the usual one (in the algebraic case this is the correct group). We will also see that for a field k the opposite of the category of finite étale k-algebras together with the functor of points of the separable closure k^s defines a Galois category. To make sense of all this we first need a good notion of a covering space of a scheme X. Let us first recall first the definition of an étale morphism:

Definition 3.1. Let $f : X \to S$ be a finitely presented map of schemes. Then f is called *étale* if it satisfies any of the following equivalent conditions:

- f is smooth and unramified
- f has the the lifting property with respect to infinitesimal extensions over S, i.e. given a first order thickening $Z_0 \to Z$ over S and a commutative diagram:

$$\begin{array}{ccc} Z_0 & \xrightarrow{u_0} & X \\ \downarrow & \overset{u}{\nearrow} & \downarrow f \\ Z & \longrightarrow & S \end{array}$$

  there exists a unique lift $u : Z \to X$.
- f is flat and unramified

We call f an (étale) *covering* if f is étale and finite. We write $\mathcal{F}\text{ét}_S$ for the category of finite étale schemes over S. Note that the property of being a covering is stable under composition and base change.

Lemma 3.2. *Let $X \xrightarrow{f} Y$ be an étale covering, then f is both open and closed. In particular we have that $Y \cong f(X) \sqcup (Y \setminus f(X))$.*

Proof. Indeed by the going-up theorem of commutative algebra finite maps are closed. Furthermore as a consequence of Chevalley's theorem one can show that finitely presented, flat morphisms are open, see [4, Section 28.24.9]. □

Definition 3.3. Let $X \xrightarrow{f} S$ be an étale covering and let S be connected. Then since f is finite and flat for any $s \in S$ we have that $(f_*\mathcal{O}_X)_s$ is a finitely generated free $\mathcal{O}_{S,s}$ module. Since the rank of a vector bundle is locally constant and S is connected we have that $\mathrm{rnk}_{\mathcal{O}_{S,s}}(f_*\mathcal{O}_X)_s \in \mathbb{N}$ does not depend on $s \in S$. We call this number the *degree* of the covering f and denote it as $\deg(f)$.

Proposition 3.4. *Let $X \to Y$ be a finitely presented morphism of schemes and $Z \to Y$ be faithfully flat. Then $X \to Y$ is étale if and only if the base change $X \times_Y Z \to Z$ is étale.*

Proof. We need to show that the properties of being flat and unramified may be checked after a faithfully flat base change. Indeed consider a pushout of rings:

$$\begin{array}{ccc} A' & \longrightarrow & B' = A' \otimes_A B \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

Where the map $A \to B$ and consequently also the map $A' \to B'$ is faithfully flat. Furthermore consider a short exact sequence of A-modules:

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

Now assume that B' is flat over B and consequently also flat over A. Then we get another exact sequence:

$$0 \longrightarrow M_1 \otimes_A B' \longrightarrow M_2 \otimes_A B' \longrightarrow M_3 \otimes_A B' \longrightarrow 0$$

Now since we have $- \otimes_A B' \cong (- \otimes_A A') \otimes_{A'} B'$ and B' is faithfully flat over A' the sequence:

$$0 \longrightarrow M_1 \otimes_A A' \longrightarrow M_2 \otimes_A A' \longrightarrow M_3 \otimes_A A' \longrightarrow 0$$

is exact as well i.e. A' is flat over A as desired. The first assertion then follows by checking it locally. Now the map $X \to Y$ is unramified if and only if the associated sheaf of Kähler differentials $\Omega^1_{X/Y}$ vanishes and again this can be checked locally. However in the affine setting as above the claim follows immediately from the identity $\Omega^1_{B/A} \otimes_A A' \cong \Omega^1_{B'/A'}$ □

Note that a finite étale morphism is faithfully flat if and only if it is surjective i.e. surjective on underlying sets.

Proposition 3.5. *Let $X \to S$ be an étale covering and S be connected. Then there exists a surjective étale covering $S' \to S$ such that the base change $X \times_S S' \to S'$ is isomorphic to a trivial covering i.e. a finite disjoint union $S' \sqcup \cdots \sqcup S' \to S'$.*

Proof. Let $X \xrightarrow{f} S$ be an étale covering. We work by induction on $n = \deg(f)$. Indeed being of degree 1 says precisely that f is an isomorphism on stalks and thus an isomorphism. Now consider the diagonal map $X \to X \times_S X$. Since f is both finite and unramified this is both a closed and open immersion. Consequently there exists some scheme X' such that $X \times_S X \cong X \sqcup X'$. Then since we have:

$$n = \deg(X \xrightarrow{f} S) = \deg(X \times_S X \to X) = \deg(X \xrightarrow{id} X) + \deg(X' \to X) = 1 + \deg(X' \to X)$$

It follows that $\deg(X' \to X) = n - 1$ and thus we can apply the induction hypothesis to get a surjective étale covering $S' \to X$ that trivializes $X' \to X$. Then the composition

$S' \to X \xrightarrow{f} S$ is the map we are looking for. Indeed since S is connected f is surjective by Lemma 3.2 so the composition is a surjective étale covering. Furthermore we have:

$$\begin{aligned} S' \times_S X &\cong (S' \times_X X) \times_S X \\ &\cong S' \times_X (X \times_S X) \\ &\cong S' \times_X (X' \sqcup X) \\ &\cong (S' \times_X X') \sqcup (S' \times_X X) \\ &\cong (S' \sqcup \cdots \sqcup S') \sqcup S' \end{aligned}$$

This completes the proof. □

We shall use this theorem later on but it also has a nice geometric interpretation: Recall that a covering map of topological spaces $p : X' \to X$ is locally trivial in the sense that any point $x \in X$ has an open neighbourhood $U \subseteq X$ such that $p^{-1}(U) \cong U \times p^{-1}(x)$. Of course $p^{-1}(U)$ is just the pullback of p with the inclusion map $U \hookrightarrow X$ that is $p^{-1}(U) \cong X' \times_X U$. Now if we think of an étale map of schemes $S' \to S$ as an 'étale open set' of S the above proposition tells us precisely that our coverings are 'étale locally' trivial. This provides some justification for the somewhat ad hoc use of étale maps as a replacement for the covering maps of topology.

Proposition 3.6. *Let $f : X \to Y$ be a morphism in $\mathcal{F}\text{ét}_S$, then f is an étale covering.*

Proof. We have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \swarrow & \\ S & & \end{array}$$

Now since closed immersions are finite, the map $Y \to S$ is separated and the composition $X \xrightarrow{f} Y \to S$ is finite, we have that f is finite as well. Furthermore since $Y \to S$ is unramified and the composition $X \xrightarrow{f} Y \to S$ is étale it follows that f is étale. □

3.2. **Verifying the axioms.** Having gotten comfortable with étale coverings we now want to show that for a connected, locally noetherian scheme S the category $\mathcal{F}\text{ét}_S$ of finite étale schemes over S is a Galois category for a suitable choice of fibre functor.

Corollary 3.7. *Let $X \xrightarrow{f} Y$ be a map in $\mathcal{F}\text{ét}_S$, then we get a factorization in $\mathcal{F}\text{ét}_S$ as in axiom* (iii) *for Galois categories via:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \twoheaddownarrow & \nearrow & \\ f(X) & & \end{array}$$

Proof. It is clear that open immersions are monomorphisms of schemes. Thus by Lemma 3.2 all that remains to show is that the map $X \to f(X)$ is a strict epimorphism. Indeed faithfully flat ring maps are injective and it is easy to see that injective maps are strict monomorphisms in the category of rings. Thus we can check locally that the relevant diagram is a pushout. □

Remark 3.8. From these considerations we also see that an étale covering is connected in the abstract categorical sense if and only if it is connected as a topological space. Furthermore on sees that the strict epimorphism in $\mathcal{F}\text{ét}_S$ are precisely the surjective maps.

Now our category $\mathcal{F}\text{ét}_S$ appears to have a decent chance of being a Galois category. But how should we construct a fibre functor? To answer this question let k be field. It is not hard to show that any étale covering of $\operatorname{Spec}(k)$ must be a finite union $\coprod_i \operatorname{Spec}(L_i)$ where each $k \subset L_i$ is a finite separable field extension. This tells us that the coverings of a field are uninteresting in terms of the Zariski topology (as they should be since $\operatorname{Spec}(k)$ is a point) and thus contain purely algebraic information. Furthermore if k is algebraically closed the category $\mathcal{F}\text{ét}_{\operatorname{Spec}(k)}$ is equivalent to the category of finite sets, i.e the coverings become entirely trivial. This elementary observation will be the key to defining our fibre functor.

Definition 3.9. Let X be a scheme. A geometric point $\bar{x} \to X$ of X is a map $\operatorname{Spec}(k(\bar{x})) \to X$ where $k(\bar{x})$ is some algebraically closed field. Furthermore if $f : Y \to X$ is a map of schemes then the geometric fibre of f at $\bar{x}$ is just the usual fibre product:

$$\begin{array}{ccc} f^{-1}(\bar{x}) & \longrightarrow & \operatorname{Spec}(k(\bar{x})) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Note that in particular finite morphisms have finite geometric fibres.

Definition 3.10. Let $S \to T$ be a map of schemes, then we define the base change functor $\mathcal{F}\text{ét}_T \to \mathcal{F}\text{ét}_S$ via pullback i.e. by mapping:

$$(X \to T) \mapsto (Z \times_T S \to S)$$

Now if $\bar{x} \to S$ is a geometric point we define the fibre functor associated to $\bar{x}$ as the base change $F_{\bar{x}} : \mathcal{F}\text{ét}_S \to \mathcal{F}\text{ét}_{\operatorname{Spec}(k(\bar{x}))} \cong \mathcal{E}\text{ns}_{\text{fin}}$. Explicitly given the pullback diagram:

$$\begin{array}{ccc} f^{-1}(x) & \longrightarrow & \operatorname{Spec}(k(x)) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

we define $F_{\bar{x}}(X)$ as the underlying set $|f^{-1}(\bar{x})| \in \mathcal{E}\text{ns}_{\text{fin}}$.

Note that this is completely analogous to the construction in topology, the key point being that $\operatorname{Spec}(k(\bar{x}))$ and the one-point space have only trivial coverings in the respective sense. Furthermore by Nakayama's Lemma we see that the degree of an étale covering of some connected scheme S is precisely the cardinality of any geometric fibre. Our goal will now be to show that $F_{\bar{x}}$ actually defines a fibre functor for $\mathcal{F}\text{ét}_S$. First some observations:

Remark 3.11. The category $\mathcal{F}\text{ét}_S$ has all finite limits, i.e. the terminal cover $S \xrightarrow{\text{id}} S$ and fibre products are computed as usual since being finite étale is stable under base change. Furthermore the empty scheme is the initial covering and finite coproducts are given by the disjoint union of schemes. It is immediate that the base change functor $\mathcal{F}\text{ét}_S \to \mathcal{F}\text{ét}_T$ commutes with these constructions for any map of schemes $T \to S$.

By the axioms we also want $F_{\bar{x}}$ to reflect isomorphisms for any geometric point $\bar{x} \to S$. This is clearly only possible if S is connected as a topological space, since $F_{\bar{x}}$ can only tell us about what is happening over the point $\bar{x}$. If we add the assumption that our base scheme is connected we get:

Proposition 3.12. *Let S be a connected scheme and $\bar{x} \to S$ be a geometric point. Then $F_{\bar{x}}$ reflects isomorphisms.*

Proof. Let $X \xrightarrow{f} Y$ be a map in $\mathcal{F}\text{ét}_S$ such that the induced set map $F_{\bar{x}}(X) \to F_{f\bar{x}}(Y)$ is an isomorphism. We may assume that Y is connected. Then since $X \to Y$ is finite étale it is an isomorphism if and only if it is locally free of rank 1 and all we need to show is the rank 1 condition. By assumption this is true in a neighbourhood of any point in the fibre over $\bar{x}$ in Y. However since the rank is locally constant and Y is connected the claim follows. □

In general we have the following definition:

Definition 3.13. Let S be a scheme and $\bar{x}, \bar{y} \to S$ be geometric points of S. Then a natural transformation $F_{\bar{x}} \to F_{\bar{y}}$ is called a *geometric path* from $\bar{x}$ to $\bar{y}$.

It will follow a posteriori from Proposition 2.25 that for a connected locally noetherian scheme any two geometric points can be connected by a geometric path.
It remains to consider quotients by finite group actions. In the following let G be a finite group. Denote by Aff_S the category of S-schemes X such that the map $X \to S$ is affine and by $\mathcal{O}_S\,\mathrm{Alg}$ the category of quasi-coherent $\mathcal{O}_S$-algebras. Recall that we have an equivalence of categories:

$$\mathrm{Aff}_S^{\mathrm{op}} \to \mathcal{O}_S\,\mathrm{Alg} \quad ; \quad (X \xrightarrow{f} S) \mapsto f_*\mathcal{O}_X$$

With pseudo-inverse given by the relative Spec construction, i.e. mapping some quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$ to the affine S-scheme $\underline{\mathrm{Spec}}_S(\mathcal{A})$. Note that our covering maps are, by definition, finite and thus affine. The idea is that colimits of schemes are in this case easier to construct as limits in the category of algebras. More to the point:

Proposition 3.14. *Let $X \xrightarrow{f} S$ be affine and consider some action $G \curvearrowright X$ over S. Then the quotient X/G exists in* Aff_S*.*

Proof. Let $\mathcal{A} = f_*\mathcal{O}_X$ be the corresponding quasi-coherent algebra and consider the induced action of G on $\mathcal{A}$. Then define:

$$\mathcal{A}^G := \ker\left(\mathcal{A} \xrightarrow{(a-ga)} \prod_{g\in G} \mathcal{A}\right)$$

On any open set $U \subset S$ this is clearly just the familiar ring of G-invariants $\mathcal{A}(U)^G$, as is expected when constructing the dual of a quotient. Since G is finite $\prod_{g\in G}\mathcal{A}$ is quasi-coherent and consequently so is $\mathcal{A}^G$. Thus the latter corresponds to a scheme $\underline{\mathrm{Spec}}_S(\mathcal{A}) \in \mathrm{Aff}_S$ and it is clear that this is the quotient X/G as desired. □

What remains to show is that if $X \to S$ is finite étale then so is $X/G \to S$. To see this we need the following lemma:

Lemma 3.15. *Let $X \to S$ be affine and $Y \to S$ be affine and flat. Given an action $G \curvearrowright X$ over S there is a natural action of G on $X \times_S Y$ such that we have $(X \times_S Y)/G \cong (X/G) \times_S Y$.*

Proof. The action is induced by the natural map $X \to X \times_S Y$ i.e. on points G acts on the first coordinate: $g(x, y) := (gx, y)$. Consequently we see that the projection map $X \times_S Y \to (X/G) \times_S Y$ is invariant under the action of G and we get an induced map $(X \times_S Y)/G \to X/G \times_S Y$. We need to show that this map is an isomorphism. This can be checked locally and since all maps are affine we may assume that all the spaces are too. Thus let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Consider the exact sequence as above:

$$0 \to A^G \to A \xrightarrow{(a-ga)} \prod_{g\in G} A$$

Since B is flat over R tensoring with B gives another exact sequence:

$$0 \to A^G \otimes_R B \to A \otimes_R B \to (\prod_{g \in G} A) \otimes_R B \cong \prod_{g \in G} (A \otimes_R B)$$

Which says precisely that $A^G \otimes_R B \cong (A \otimes_R B)^G$. This proves the claim. □

Proposition 3.16. *Let $X \to S$ be a covering and let G act on X. Furthermore let S be locally noetherian, then the quotient X/G exists in $\mathcal{F}\text{ét}_S$. Furthermore for any geometric point $\bar{x} \to S$ the fibre functor $F_{\bar{x}}$ commutes with these quotients.*

Proof. It suffices to show that the affine map $X/G \to S$ constructed above is finite étale. This can be checked locally so we again assume that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ where R is noetherian. Thus our claim is that the map $R \to A^G$ is finite étale. Indeed it is finite since R is noetherian and A^G is by construction a submodule of a finitely generated R-module. Now by Proposition 3.5 there exists some finite, étale, faithfully flat R-algebra B such that the base change $A' := A \otimes_R B$ is isomorphic to a finite product $B \times \cdots \times B$. Now by Proposition 3.4 we can check the étale property after a faithfully flat base change. Furthermore we know by Lemma 3.15 that such a base change commutes with taking finite quotients. Thus it suffices to show that $(A')^G$ is an étale B-algebra. However since any B-automorphism of $A' = B \times \cdots \times B$ must necessarily be permutation of the copies of B. It follows that the ring of invariants $(A')^G$ is also a finite product of copies of B and thus étale. Now let $\bar{x} \to S$ be some geometric point and consider the fibre functor $F_{\bar{x}}$. As before we can make a faithfully flat base change to a trivial covering $X' \to S'$. In this setting it is clear that the natural map $F_{\bar{x}}(X')/G \to F_{\bar{x}}(X'/G)$ is an isomorphism. Now, as we have already noted, taking fibres commutes with fibre products and whether a map is an isomorphism can clearly be checked after a faithfully flat base change. Thus the second claim follows. □

The only thing left to show is that $F_{\bar{x}}$ maps essential epimorphisms to surjective set-maps. This follows from the explicit construction of the factorization as in Lemma 3.2, in particular the fact that the essential epimorphisms in $\mathcal{F}\text{ét}_S$ are precisely the surjective maps.
In summary we have proven our second main theorem:

Theorem 3.17. *Let S be a locally noetherian and connected scheme. Then for any geometric point $\bar{x} \to S$ the category $\mathcal{F}\text{ét}_S$ of étale coverings of S together with the base change functor $F_{\bar{x}} \to \mathcal{E}ns_{\text{fin}}$ defines a Galois category. In particular we get an equivalence of categories:*

$$\mathcal{F}\text{ét}_S \xrightarrow{\sim} \mathcal{E}ns_{\text{fin}}^{\mathrm{Aut}(F_{\bar{x}})} \quad ; \quad (X \to S) \mapsto F_{\bar{x}}(X) = X \times_S \mathrm{Spec}(k(\bar{x})) \tag{8}$$

Definition 3.18. Let S be a locally noetherian and connected scheme. Furthermore let $\bar{x} \to S$ be a geometric point. Then the *étale fundamental group* $\pi_1(S, \bar{x})$ of S with respect to $\bar{x}$ is defined as the profinite group $\pi_1(\mathcal{F}\text{ét}_S, F_{\bar{x}}) = \mathrm{Aut}(F_{\bar{x}})$. The *fundamental groupoid* $\Pi_1(S)$ is defined as $\Pi_1(\mathcal{F}\text{ét}_S)$.

Remark 3.19. It is clear from our discussion that the isomorphism type of $\pi_1(S, x)$ does not depend on x and the isomorphisms are unique up to inner automorphisms. Equivalently the fundamental groupoid $\Pi_1(S)$ is a connected groupoid. As usual we will sometimes suppress the choice of base point in our notation and just write $\pi_1(S)$.

Corollary 3.20. *All finite limits and colimits of finite étale schemes over S exist in $\mathcal{F}\text{ét}_S$.*

Proposition 3.21. *The fundamental group is functorial in the following sense: Given a map of schemes $f : T \to S$ and a geometric point $\bar{x} \to T$ if we let $\bar{y} = f\bar{x}$ we get a group* *homomorphism $f_* : \pi_1(T, \bar{x}) \to \pi_1(S, \bar{y})$ inducing a commutative diagram:*

$$\begin{array}{ccc} \mathcal{F}\text{ét}_S & \longrightarrow & \mathcal{F}\text{ét}_T \\ \downarrow & & \downarrow \\ \mathcal{E}ns_{\text{fin}}^{\pi_1(S,\bar{y})} & \longrightarrow & \mathcal{E}ns_{\text{fin}}^{\pi_1(T,\bar{x})} \end{array}$$

Where $\mathcal{F}\text{ét}_S \to \mathcal{F}\text{ét}_T$ is just the base change defined earlier. Furthermore this induces a functor $\Pi_1(T) \to \Pi_1(S)$.

Proof. This is a special case of Proposition 2.26. □

We have thus constructed an analogue of the topological fundamental group for schemes, if one accepts the fact that finite étale maps are the correct notion of covering space. However we also promised that this theory would generalize Galois theory and indeed it does.

Theorem 3.22. *Let k be a field and let $\bar{x} \to \text{Spec}(k)$ be a geometric point i.e. a choice of algebraic closure $\bar{k} = k(\bar{x})$ of k. Let $k^s \subseteq \bar{k}$ be the separable closure of k in $\bar{k}$ Then there exists a natural isomorphism $\pi_1(\text{Spec}(k), \bar{x}) \cong \text{Gal}(k^s/k)$. More to the point the fibre functor $F_{\bar{x}} : \mathcal{F}\text{ét}_{\text{Spec}(k)} \to \mathcal{E}ns_{\text{fin}}$ is prorepresented by $\text{Spec}(k^s)$.*

Proof. Since by the Yoneda lemma we have:

$$\begin{aligned} \text{Aut}(\text{Hom}_k(\text{Spec}(k^s), -)) &= [\text{Hom}_k((\text{Spec}(k^s), -), \text{Hom}_k(\text{Spec}(k^s, -)] \\ &\cong \text{Hom}_k(\text{Spec}(k^s), \text{Spec}(k^s)) \\ &\cong \text{Hom}_k(k^s, k^s) \\ &= \text{Aut}(k^s/k) \end{aligned}$$

it suffices to prove the second assertion. Let $X/\text{Spec}(k)$ be finite étale. We may assume that X is connected, then $X = \text{Spec}(L)$ where $k \subseteq L \subseteq k^s$ is some separable extension of finite degree d. Then we have

$$F_{\bar{x}}(\text{Spec}(K)) = \text{Spec}(L \otimes_k \bar{k}) \cong \text{Spec}(\prod_{1 \le i \le d} \bar{k}) \cong \coprod_{1 \le i \le d} \text{Spec}(\bar{k})$$

Where each copy of $\text{Spec}(\bar{k})$ corresponds to a way to embed K into the algebraic closure $\bar{k}$ over k. Since K/k is separable this corresponds to an embedding into the separable closure k^s over k i.e. a commutative diagram:

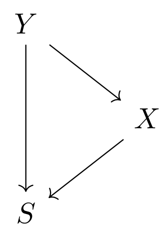
$$\begin{array}{ccc} \bar{k} & & \\ \uparrow & \nwarrow & \\ k^s & \longleftarrow & L \\ \uparrow & \nearrow & \\ k & & \end{array}$$

Where the embedding $k \to \bar{k}$ was chosen beforehand. However this is precisely an element of $\text{Hom}_k(\text{Spec}(k^s), \text{Spec}(L)) = \text{Hom}_k(L, k^s)$. Thus we have a natural isomorphsim $\text{Hom}(\text{Spec}(k^s), -) \cong F_{\bar{x}}$ as claimed. □

Remark 3.23. Note that fields are simply connected if and only if they are separably closed. This reflects the fact that, while they have no underlying point-set topological structure, they are not trivial spaces in terms of the étale topology. This is also a good time to reflect on

the Galois objects introduced earlier. Indeed we have seen that $X \in \mathcal{F}\text{ét}_{\text{Spec}(k)}$ is Galois iff it is connected and we have $\# \operatorname{Aut}(X) = \# F_{\bar{x}}(X)$. In our case this says precisely that $X = \operatorname{Spec}(L)$ where L/k is finite and separable such that $\# \operatorname{Aut}(L/k) = [L : k]$ which is the usual notion of a Galois extension of fields. Furthermore if we let $L = {}^{k[x]}/_{f}$ for some $f \in k[x]$ irreducible we see that the fibre of the map $\operatorname{Spec}(L) \to \operatorname{Spec}(k)$ is given by $\operatorname{Hom}_k(L, k^s)$ which is in natural bijection with the roots of f. In particular the purely algebraic fact that $\operatorname{Aut}(L/k)$ permutes the roots of f has the geometric interpretation of a deck transformation acting on the fibres of a covering. In particular a covering is Galois if its automorphism group acts transitively on the fibres.
We also see that even in the simple case of fields the universal covering need not exist in the strict sense. The subtlety here is that the space k^s does indeed prorepresent the fibre functor, however the map $k \to k^s$ is in general not a covering in our sense because it is not finite. More to the point a universal covering exists in the strict sense if and only if the étale fundamental group is finite.

Corollary 3.24 (Galois Closure)**.** *Let S be a connected locally noetherian scheme and X/S be finite étale. Then X is dominated by some Galois cover Y/S i.e. we have a commutative diagram:*



Proof. This is exactly Theorem 2.19 □

In the case of a $S = \operatorname{Spec}(k)$ and a finite separable field extension L/k this tells us that there exists some finite separable extension L'/L such that L'/k is Galois. Moreover it follows from the fact that Galois categories are artinian (Corollary 2.11) that we can find a Galois object which is minimal with this property. This recovers the usual notion of Galois closure.

3.3. **The short exact sequence.** Even when working over a field figuring out the fundamental group of a scheme can be arbitrarily difficult. However in a sense we can split up the problem into a geometric and an arithmetic one, the former stemming from the 'genuine' geometric structure that remains when base changing to an algebraically closed field and the latter stemming from the potentially nontrivial Galois group of the ground field. In particular when working over $\mathbb{C}$ the étale fundamental group should be purely 'geometric' and agree with the profinite completion of the topological fundamental group of the associated complex manifold. Proving comparison theorems like this is outside the scope of this thesis but we will make the first notion precise. Our goal will be to prove the following:

Theorem 3.25. *Let k be a field, S a locally noetherian connected k-scheme. Denote by $\bar{k}$ the algebraic closure and by $S_{\bar{k}}$ the base change to $\operatorname{Spec}(\bar{k})$. Then there exists a natural exact sequence:*

$$1 \longrightarrow \pi_1(S_{\bar{k}}) \longrightarrow \pi_1(S) \longrightarrow \pi_1(\operatorname{Spec}(k)) \longrightarrow 1$$

Where the maps are induced via functoriality from the bottom path in the pullback diagram:

$$\begin{array}{ccc} S_{\bar{k}} & \longrightarrow & \mathrm{Spec}(\bar{k}) \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

Note that by our previous discussion $\pi_1(\mathrm{Spec}(k))$ is just the absolute Galois group of k. We follow the exposition given in [4, Section 54.4] and [4, Section 54.14]. To prove this theorem we need to return to the abstract theory of Galois categories for a while. More precisely: Given Galois categories $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ with fundamental groups π, π', π'' and a commutative diagram:

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C}'' \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{E}\mathrm{ns}_{\mathrm{fin}} & & \end{array}$$

Just as in Proposition 2.26 we consider the associated commutative diagram:

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C}'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}\mathrm{ns}^{\pi}_{\mathrm{fin}} & \longrightarrow & \mathcal{E}\mathrm{ns}^{\pi'}_{\mathrm{fin}} & \longrightarrow & \mathcal{E}\mathrm{ns}^{\pi''}_{\mathrm{fin}} \end{array}$$

Along with the sequence $\pi'' \to \pi' \to \pi$ inducing it. We want to understand this sequence in terms of the functors $\mathcal{C} \to \mathcal{C}' \to \mathcal{C}''$ which, by our diagram, are equivalent to the functors $\mathcal{E}\mathrm{ns}^{\pi}_{\mathrm{fin}} \to \mathcal{E}\mathrm{ns}^{\pi'}_{\mathrm{fin}} \to \mathcal{E}\mathrm{ns}^{\pi''}_{\mathrm{fin}}$. Thus for our current purposes we can identify $\mathcal{C}$ with $\mathcal{E}\mathrm{ns}^{\pi}_{\mathrm{fin}}$ and so on.

Proposition 3.26. *The composition $\pi'' \to \pi$ is trivial if and only if for any $X \in \mathcal{C}$ the image under the composition $\mathcal{C} \to \mathcal{C}''$ is a finite coproduct of terminal objects.*

Proof. This says precisely that the morphism of profinite groups $\pi'' \to \pi$ is the constant map if and only if each for each continuous action of π on some finite set S the induced action of π'' on S is trivial. Indeed if $1 \neq g \in \pi$ lies in the image of π'' let $\tau \subseteq \pi$ be some open subgroup not containing g. Then π'' acts nontrivially on π/τ. The other implication is clear. □

Proposition 3.27. *Let α be the smallest closed normal subgroup containing the image of $\pi'' \to \pi'$ and let $\beta = \ker(\pi' \to \pi)$. Furthermore let $\mathcal{C} \xrightarrow{H} \mathcal{C}' \xrightarrow{H'} \mathcal{C}''$ as above such that the composition is trivial. Then we have $\alpha = \beta$ if for each $X' \in \mathcal{C}'$ such that $H'(X')$ is a finite coproduct of terminal object there exists some $X \in \mathcal{C}$ and an epimorphism $H(X) \to X'$.*

Proof. First of all we already know that $\alpha \subseteq \beta$ thus we only need to show the other inclusion. In terms of group actions our condition says the following: Given a π'-set X' such that the induced action of π'' is trivial there exists a π-set X and a surjection $X \to X'$ of π'-sets. In particular we see that the π'-action on X' factors through π. Now if $\gamma \subseteq \pi'$ is some open subgroup then the action of π' on $\pi'/\gamma\alpha$ factors through π. It follows that $\gamma\alpha \subseteq \beta$ for any open γ. Then since α is closed we have $\beta \subseteq \alpha$ as desired. □

Proposition 3.28. *The image of the map $\pi'' \to \pi'$ is normal if and only if whenever $X' \in \mathcal{C}'$ is a connected object that admits a map $*'' \to H(X')$ then $H(X')$ is a finite coproduct of terminal objects.*

Proof. The second statement translates to the following: Given some open subgroup $\tau' \subseteq \pi'$ such that π'' fixes some $g\tau' \in \pi'/\tau'$ then π'' fixes all elements of π'/τ'. Equivalently if $\operatorname{im}(\pi'' \to \pi')$ is contained in τ' then so is every conjugate of it. However this holds for arbitrary open τ' if and only if $\operatorname{im}(\pi'' \to \pi')$ is normal. □

Proposition 3.29. *The map $\pi' \to \pi$ is surjective iff the functor $\mathcal{C} \to \mathcal{C}'$ is fully faithful.*

Proof. The 'only if' part is easy: Pulling back along a surjective map is always injective. Furthermore if $\pi' \to \pi$ is surjective any set-map of π-sets is π'-invariant iff it is π-invariant. Conversely suppose that $\pi^p \to \pi$ is not not surjective. Then if $\pi = \varprojlim \pi_i$ there exists some index i such that the induced map $\pi' \to \pi_i$ is not surjective. Then the product topology allows us to construct a proper open subgroup $\tau \subseteq \pi$ containing the image of π'. Then π' acts trivially on the finite set $X = \pi/\tau$ but π does not (unless π itself is trivial). Thus the map $\operatorname{Hom}_\pi(X, X) \to \operatorname{Hom}_{\pi'}(X, X)$ is not bijective i.e. our functor is not fully faithful. □

This is particularly useful along with the following characterization of fully faithful functors between Galois categories:

Lemma 3.30. *Let $F : \mathcal{C} \to \mathcal{C}'$ as above. Then the following are equivalent:*

(1) F is fully faithful
(2) For any $X \in \mathcal{C}$ the map $F : \operatorname{Hom}_{\mathcal{C}}(*, X) \to \operatorname{Hom}_{\mathcal{C}'}(*', F(X))$ is a bijection.
(3) For any connected $X \in \mathcal{C}$ the map $F : \operatorname{Hom}_{\mathcal{C}}(*, X) \to \operatorname{Hom}_{\mathcal{C}'}(*', F(X))$ is a surjection.
(4) F maps connected objects to connected objects.

Proof. The implications 1. ⇒ 2. ⇒ 3. are clear. Note that a map $* \to X$ is precisely a fixed point of our π-action. For 3. ⇒ 4. suppose we have a π-set X such that π acts transitively on X but π' has multiple orbits. Let Y be one of them and consider the π-set $X/Y = (X \setminus Y) \sqcup *$. Then π still acts transitively on X/Y i.e. we have a connected object of $\mathcal{C}$ however the action of π' has a new fixed point by construction i.e. 3. does not hold as desired. Finally let τ be as in the previous proof and choose some nontrivial orbit X of the π-action on π/τ. Then X is connected in $\mathcal{C}$ but not in $\mathcal{C}'$. This proves 4. ⇒ 1. and completes the proof. □

Proposition 3.31. *Let $H : \mathcal{C}' \to \mathcal{C}''$ and $h : \pi'' \to \pi'$ as above. The map $\pi'' \to \pi'$ is injective if and only if for every connected $X'' \in \mathcal{C}''$ there exists some $X' \in \mathcal{C}'$, $Y'' \in \mathcal{C}''$ and a diagram:*

$$\begin{array}{ccccc} & & Y'' & & \\ & \swarrow & & \searrow & \\ H(X') & & & & X'' \end{array}$$

Where the left-hand map is monic and the right-hand map is epic.

Proof. Suppose h is injective and let $\tau'' \subseteq \pi''$ be an open subgroup. Then since h is continuous there exists some open subgroup $\tau' \subseteq \pi'$ such that $h^{-1}(\tau') \subseteq \tau''$. Then we get the desired diagram in $\mathcal{C}''$:

$$\begin{array}{ccccc} & & \pi''/h^{-1}(\tau'') & & \\ & \swarrow & & \searrow & \\ \pi'/\tau' & & & & \pi''/\tau'' \end{array}$$

Conversely consider some $g \in \ker(h)$. Then for any open subgroup τ'' and the quotient π''/τ'' we can find some π''-set Y'' and some π'-set X' as above. Then since g is killed by h it acts trivially on X'. Then since $Y'' \to H(X')$ is monic i.e. in particular an injective set-map g

also acts trivially on Y''. Furthermore $Y'' \to X''$ is surjective as a set-map so g acts trivially on π''/τ''. Since τ'' was arbitrary it follows that $g = 1$ and thus $\ker(h)$ is trivial. □

Now we are in a position to prove Theorem 3.25 :

Proof. We consider the maps $S_{\bar{k}} \to S \to \mathrm{Spec}(k)$ and the induced sequence:

$$1 \to \pi_1(S_{\bar{k}}) \to \pi_1(S) \to \pi_1(\mathrm{Spec}(k)) \to 1$$

- To make our life easier we assume that k is a perfect field such that $\bar{k} = k^s$. This is legitimate because the base change to the perfection of k induces an isomorphism on fundamental groups. For a proof see [4, Section 54.14.2].
- Note that for any X/k finite étale the base change along the composition is clearly a trivial covering of $S_{\bar{k}}$. Thus by Proposition 3.26 the composition $\pi_1(S_{\bar{k}}) \to \pi_1(k)$ is trivial.

- Now let X/k be connected i.e. $X = \mathrm{Spec}(L)$ for some finite separable field extension L/k. Then the map $S_{\bar{k}} \to S_L$ is surjective. However by assumption $S_{\bar{k}}$ is connected so the base change S_L is as well. Thus by Proposition 3.29 the map $\pi_1(S) \to \pi_1(\mathrm{Spec}(k))$ is a surjection.
- Let $X \to S$ be a connected étale covering such that the base change $X \times_S S_{\bar{k}} = X_{\bar{k}} \to S_{\bar{k}}$ admits a morphism $* \to X_{\bar{k}}$ in $\mathcal{F}\text{ét}_{S_{\bar{k}}}$ i.e. a section $S_{\bar{k}} \to X_{\bar{k}}$. We need to show that $X_{\bar{k}}$ is a finite disjoint union of copies of $S_{\bar{k}}$. We have seen that étale coverings are open and thus $s(S_{\bar{k}})$ is a connected component of $X_{\bar{k}}$. We need more sections and we get them as follows: Consider some map $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and the induced map of k-schemes $\sigma : \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(\bar{k})$. Taking the fibre product twice gives a commutative diagram:

$$\begin{array}{ccccc} X_{\bar{k}} & \longrightarrow & S_{\bar{k}} & \longrightarrow & \mathrm{Spec}(\bar{k}) \\ \downarrow{\scriptstyle \sigma''} & & \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ X_{\bar{k}} & \longrightarrow & S_{\bar{k}} & \longrightarrow & \mathrm{Spec}(\bar{k}) \end{array}$$

  Then letting s^σ be the composition $S_{\bar{k}} \xrightarrow{\sigma} S_{\bar{k}} \xrightarrow{s} X_{\bar{k}} \xrightarrow{(\sigma'')^{-1}} X_{\bar{k}}$ we get a new section of $X_{\bar{k}} \to S_{\bar{k}}$. We claim that:

$$X_{\bar{k}} = \bigcup_{\sigma \in \mathrm{Gal}(\bar{k}/k)} s^\sigma(S_{\bar{k}})$$

  Indeed since étale coverings are finite maps the right hand side is in fact a finite union of connected components. Since it is also stable under the action of the Galois group $\mathrm{Gal}(\bar{k}/k)$ it can be shown that it is the preimage along the base change map $X_{\bar{k}} \to X$ of some closed set $T \subseteq X$ (see [4, Section 32.7.10]). However since the map $X \to k$ is universally open we see that T is also open. Thus since we assumed that X was connected we have $T = X$ and the claim follows. Then by Proposition 3.28 we see that the image of $\pi_1(S_{\bar{k}}) \to \pi_1(S)$ is normal.
- We show that every étale covering $\bar{X} \to S_{\bar{k}}$ embeds as an open subscheme of the base change of some étale covering of S and use Proposition 3.31 to conclude that $\pi_1(S_{\bar{k}}) \to \pi_1(S)$ is injective. Thus let $\bar{X} \to S_{\bar{k}}$ be some covering. Then since we have:

$$\mathrm{Spec}(\bar{k}) = \varprojlim_{\substack{L/k \\ \text{finite} \\ \text{separable}}} \mathrm{Spec}(L)$$

we can find some finite separable extension L/k and some finite étale covering $X_L \to S_L$ such that we have:

$$\bar{X} = S_{\bar{k}} \times_{S_L} X_L = \mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(L)} X_L$$

Indeed since any such covering can be glued together from finitely many schemes each defined by some finite set of polynomial equations we can find some extension L as above such that L contains all coefficients of the defining equations. Gluing these equations over L defines our covering X_L as desired. Then the composition $X_L \to S_L \to S$ is also finite étale. It suffices to show that the natural map $\bar{X} = X_L \times_{\mathrm{Spec}(L)} \mathrm{Spec}(\bar{k}) \to X_L \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ is an open and closed immersion. This follows from the fact that the multiplication map $L \otimes_k \bar{k} \to \bar{k}$ defines an open and closed immersion and that our properties are stable under base change. Thus we are done.

– Let $X \to S$ be finite étale such that the base change $X_{\bar{k}} \to S_{\bar{k}}$ is a trivial covering. To apply Lemma 3.27 we need to show that there exists some finite étale k-scheme Y such that the base change $Y \times_k S$ surjects onto X. Arguing as in the previous point we can find some finite separable extension L/k such that $X_L \cong S_L \sqcup \cdots \sqcup S_L$. Thus setting $Y = \mathrm{Spec}(L) \sqcup \cdots \sqcup \mathrm{Spec}(L)$ does the trick and it follows that the sequence is exact in the middle.

□

4. Examples of fundamental groups

We now move on to some explicit computations of étale fundamental groups. Since our universal covers have a tendency not to exist there is rarely a straightforward way to do this. Usually we have to work out directly what the Galois covers of our scheme are and consequently there is no general recipe for cooking up the fundamental group. Even worse it is not even clear what is meant by this. Indeed how does one 'calculate' the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$? Nonetheless we shall try to give some examples, first some with arithmetic and then some with geometric flavor. By Theorem 3.25 we know that in the latter case we can restrict ourselves to working over an algebraically closed field without losing much information.

4.1. **Normal schemes.** For starters we sketch some ideas about normal schemes before moving to a more in-depth discussion of elliptic curves.

Definition 4.1. A scheme X is called *normal* if for every point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is an integrally closed domain.

Note that connected normal schemes are necessarily integral. A very fruitful perspective is the following: Given an étale covering $Y \to X$ of noetherian integral schemes we instead want to consider the extension of function fields i.e. meromorphic functions $K(X) \subseteq K(Y)$. Geometrically this connects the fundamental group to the divisor class group and allows us to study it with some powerful tools such as the Riemann-Roch theorem. On the arithmetic side if we let X be some Dedekind domain we see how the fundamental group interacts with ramification theory and the like. It is thus natural to ask what the fundamental group $\pi_1(X)$ has to do with the Galois group of the function field $K(X)$. As it turns out if X is a normal scheme then $\pi_1(X)$ is a quotient of $\mathrm{Gal}(K(X)^s/K(X))$. To see this recall the notion of the normalization of a scheme: Given an integral scheme with an affine covering $\bigcup \mathrm{Spec}(A_i)$ and a field extension $L/K(X)$ take the integral closure $A_i \subset B_i$ in L. Then the $\mathrm{Spec}(B_i)$ glue to an integral scheme Y/X called the normalization of X in L. Clearly this is just a global version of integral closure and one sees easily that any map from a normal scheme to X factors through Y. One can then prove the following:

Lemma 4.2. *Let X be a connected normal scheme and $Y \to X$ be a connected étale covering. Then Y is normal and thus the normalization of X in $K(Y)$.*

Proof. [4, Section 54.11.2] □

Theorem 4.3. *Let X be a noetherian normal scheme let $K = K(X)$ be its function field and let L^{un} be the compositum of all finite field extensions L/K such that the normalization of X at L is a covering of X. Then we have $\pi_1(X) \cong \mathrm{Gal}(L^{\mathrm{un}}/K)$ and furthermore the canonical maps:*

$$\mathrm{Spec}(L^{\mathrm{un}}) \to \mathrm{Spec}(K) \to X$$

induce an exact sequence of profinite groups:

$$1 \to \mathrm{Gal}(K^s/L^{\mathrm{un}}) \to \mathrm{Gal}(K^s/K) \to \mathrm{Gal}(L^{\mathrm{un}}/K) \to 1$$

Proof. The fact that $\pi_1(X) \cong \mathrm{Gal}(L^{\mathrm{un}}/K)$ is immediate from our previous lemma. Exactness on the right-hand side then also follows from the lemma and our considerations in Proposition 3.30. The rest of the sequence is Galois theory. For details see [4, Section 54.11.3]. □

This allows us to give some examples of affine fundamental groups. First note that any étale covering of an affine scheme is necessarily affine.

Example 4.4. We have $\pi_1(\mathrm{Spec}(\mathbb{Z})) = 1$. Indeed by our considerations above any étale covering is given by some ring of integers $\mathbb{Z} \subseteq \mathcal{O}_K$ associated to some number field $\mathbb{Q} \subseteq K$. However by Minkowski's theorem any such extension is ramified about at least one prime and thus there are only trivial coverings. Furthermore for any $n \in \mathbb{Z}$ we see that $\pi_1(\mathbb{Z}[\frac{1}{n}]) = \mathrm{Gal}(\mathbb{Q}^{(n)}/\mathbb{Q})$ where $\mathbb{Q}^{(n)}$ is the compositum of all number fields unramified away from n, i.e. over the primes that do not divide n.

Example 4.5. Let p be some prime, then since we have that $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \cong \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ our theorem yields: $\pi_1(\mathbb{Z}_p) \cong \hat{\mathbb{Z}}$.

The general principle here is that if our field of meromorphic functions K is nice enough we can understand the étale coverings of our space in terms of the valuations on K. Note that, if we have a one dimensional regular scheme X such that the valuations on $K(X)$ correspond to the points on X, this is essentially fancy language for saying that we study the individual branching points of spaces finite over X. We will think about this a little more when we discuss curves next.

4.2. Curves and some affine spaces.

In the following let k denote an algebraically closed field. First another elementary example:

Example 4.6. Consider the affine space $\mathbb{A}_k^n = \mathrm{Spec}(k[t_1, \ldots, t_n])$. Then any connected covering $X \to \mathbb{A}_k^n$ is given by the vanishing set of some polynomials $f_1, \ldots, f_l \in k[t_1, \ldots, t_n]$. However since k is algebraically closed for each $(\eta_i) \in k^n$ the base change given by setting $t_i = \eta_i$ is a trivial covering. Thus the f_i all vanish at every point of $\mathrm{Spec}(k)$ and since k is algebraically closed we have $f_i = 0 \ \forall i$ i.e. the covering is trivial. It follows that we have $\pi_1(\mathbb{A}_k^n) = 1$ as one would expect.

Definition 4.7. A space X is called a *curve* if it is an integral, proper, finite type k-scheme of dimension one. A curve is called *smooth* if it is also smooth over k.

Note that a curve is smooth if and only it is regular, i.e. the stalks are all discrete valuation rings. Recall that the genus $g(X)$ of a curve X is defined as $\dim_k(H^0(\Omega_X))$ where Ω_X denotes the canonical line bundle on X. We will consider the cases of genus zero an one. Now since the theorem we wish to employ only talks about finite maps *between curves* we first need to prove that this loses no generality:

Proposition 4.8. *Let X be a smooth curve and $f : Y \to X$ be a connected étale covering, then Y is also a smooth curve.*

Proof. First note that, since finite morphisms are proper (by the going-up theorem) and properness is stable with respect to composition, Y is proper over k. Similarly one sees that Y is of finite type over k. Consider then a point $y \in Y$ and the induced map on stalks $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Then since f is an étale covering $\mathcal{O}_{Y,y}$ is a finite, flat, and unramified extension of a discrete valuation ring and hence itself a discrete valuation ring. More to the point the image $t\mathcal{O}_{Y,y}$ of a uniformizer $t \in \mathcal{O}_{X,f(y)}$ is again a uniformizer for the local ring $\mathcal{O}_{Y,y}$ since the map is unramified. In particular the stalks of Y are one dimensional integral domains and thus Y is a smooth integral scheme of dimension one. □

Thus we only need to consider finite maps between smooth curves, which is necessarily already a flat map, compare [2, Chapter III.9.4]. Then for our purposes all we need to ask ourselves is wether these maps are ramified or not. Let $X \to Y$ be such a map. We mentioned earlier that we can have ramification at any point of $x \in X$ and want to make this idea precise. Let $t \in \mathcal{O}_{Y,y}$ be a uniformizing parameter and $y = f(x)$. Furthermore let ν_x be a valuation on $\mathcal{O}_{X,x}$ and consider t as an element of this ring via the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. We define the *ramification index* of f at x as:

$$e_x := \nu_x(t)$$

For $e_x > 1$ we say that f ramifies at x and call x a branching point of the covering. If $e_x = 1$ we say that f is unramified at x and it is easy to see that f is unramified in the usual sense if and only if it is unramified at every point of X. Furthermore f is called tamely ramified at x if $\mathrm{char}(k)$ does not divide e_x. With this terminology one has the following consequence of the Riemann-Roch theorem:

Theorem 4.9 (Hurwitz)**.** *Let $f : X \to Y$ be a finite separable morphism of smooth curves which is tamely ramified. Let d be the degree of the field extension $K(X)/K(Y)$ Then we have:*

$$2g(X) - 2 = d(2g(Y) - 2) + \sum_{x \in X} (e_x - 1)$$

Where $g(X)$ denotes the genus of X and so on.

Proof. See [2, Chapter IV.2.4] □

Note that if $X \xrightarrow{f} Y$ is an étale covering of smooth curves, then $d = [K(Y) : K(X)]$ is equal to the degree of f as defined earlier. Then Theorem 4.9 tells us that:

$$2g(X) - 2 = d(2g(Y) - 2)$$

In particular since $g(\mathbb{P}^1_k) = 0$ we immediately get:

Corollary 4.10. *The projective line has only trivial coverings i.e. we have $\pi_1(\mathbb{P}^1_k) = 1$*

The case of curves with genus one is somewhat harder but also more interesting:

Definition 4.11. A k-scheme X is called an *elliptic curve* if it is a smooth curve of genus one.

Remark 4.12. Now some simple arithmetic tells us that any étale covering space of some elliptic curve is again an elliptic curve and furthermore any finite map between elliptic curves is a covering.

We now recall some basic facts:

Definition 4.13. Let $\mathcal{C}$ be a category with finite products and AbGrp be the category of abelian groups. Furthermore denote the terminal object of $\mathcal{C}$ by $* \in \mathcal{C}$. If $G \in \mathcal{C}$ is some object of $\mathcal{C}$ then an *abelian group object* structure on G is a factorization:

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\mathrm{Hom}(-,G)} & \mathrm{Set} \\ & \searrow \quad \nearrow & \\ & \mathrm{AbGrp} & \end{array}$$

where AbGrp $\to$ Set is the forgetful functor. Given such a factorization we understand $\mathrm{Hom}(-, G)$ as functor from $\mathcal{C}^{\mathrm{op}}$ to AbGrp. In particular we have a map $0 : * \to G$ called the *identity element* of G corresponding to the unique natural group homomorphism $\mathrm{Hom}(X, *) = 0 \to \mathrm{Hom}(X, G)$. A morphism of group objects is a natural transformation of the associated functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{AbGrp}$.

Proposition 4.14. *Let X be an elliptic curve over k and let $x : \mathrm{Spec}(k) \to X$ be a k-valued point. Then there exists a unique abelian group scheme structure on X such that x is the identity element.*

Proof. See [2, Chapter IV.4] □

In the following if we want to fix a group scheme structure on an elliptic curve X we simply give the associated identity element $x \in X(k)$ and also call the pair (X, x) an elliptic curve.

Definition 4.15. An *isogeny* of elliptic curves $(X, x) \to (Y, y)$ is a morphism of the associated group schemes. In particular such a morphism sends $x \mapsto y$.

Proposition 4.16. *Let X, Y be elliptic curves and let $f : X \to Y$ be a map of k-schemes. Then f is either constant or surjective. Furthermore if we fix basepoints $x \in X(k)$, $y \in Y(k)$ such that $fx = y$ then f is an isogeny.*

Note that, since we are working over an algebraically closed field, choosing a geometric point for the purpose of calculating the fundamental group is the same as choosing a basepoint as above. In particular if we have an étale covering of elliptic curves, which is then necessarily surjective, we can choose basepoints such that the fibre of the covering is precisely its kernel as a map of group schemes. Furthermore we can just work with the k-valued points of the kernel so we actually have a finite abelian group. In the following it is understood that we have chosen basepoints cleverly such that everything makes sense.

Lemma 4.17. *Let $f : X \to Y$ be an étale covering of elliptic curves X, Y and x be k-valued point of* $\ker(f)$. *Consider the addition map $\tau_x : X \to X$ given on k-valued points by $x' \mapsto x' + x$. Then the map*

$$\tau : \ker(f)(k) \to \mathrm{Aut}(X \xrightarrow{f} Y) \ ; \ x \mapsto \tau_x$$

defines an isomorphism of abelian groups. In other words the natural action of $\mathrm{Aut}(X \xrightarrow{f} Y)$ *on the fibre of f is transitive i.e. f is a Galois cover.*

Proof. It is clear that the map is an injective morphism of groups. However since the covering is connected the action of the automorphism group is always *free* so in particular we have $\# \operatorname{Aut}(X \xrightarrow{f} Y) \leq \# \ker(f)(k)$ as in Proposition 2.16. But then equality necessarily holds so the map is a bijection. □

Thus all étale covering maps of elliptic curves are already Galois. However the situation is even better. Since we are talking about group objects any isogeny has a dual map between the dual objects. Furthermore since elliptic curves are self dual we get a map in the opposite direction which trivializes our original covering in a sense. We shall not discuss this construction in detail but the precise statement as we need it is the following:

Theorem 4.18. *Let $Y \xrightarrow{f} X$ be an isogeny of degree n and consider the isogeny $\mu_n : X \to X$ given on points by multiplication by n. Then there exists an isogeny $f^* : X \to Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & & \\ & \searrow^{f^*} & \\ \downarrow{\scriptstyle \mu_n} & & Y \\ & \swarrow_{f} & \\ X & & \end{array}$$

Proof. See [3, Chapter III.6] □

Proposition 4.19. *Suppose* $\operatorname{char}(k) \nmid n$ *then* $X \xrightarrow{\mu_n} X$ *is étale and Galois.*

Proof. It is not hard to see that μ_n is finite and thus the first part follows from Remark 4.12. The second part is then a consequence of Lemma 4.17. □

It follows that for $\operatorname{char}(k) = 0$ the system of maps $(X \xrightarrow{\mu_n} X)_{n \in \mathbb{Z}}$ is cofinal in $\mathcal{F}\text{ét}_X$. Note that, since the degree is multiplicative, μ_n necessarily has degree n^2. Thus the group of deck transformations $\operatorname{Aut}(X \xrightarrow{\mu_n} X) = \ker(\mu_n)(k)$ is an abelian n-torsion group of order n^2 and consequently isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$.

Corollary 4.20. *Let X be an elliptic curve over k and* $\operatorname{char}(k) = 0$*. Then we have:*

$$\pi_1(X) \cong \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^2 \cong \hat{\mathbb{Z}}^2$$

If we take $k = \mathbb{C}$ then in terms of complex geometry the notion of an elliptic curve translates precisely to that of a one dimensional, compact, connected, complex manifold of (topological) genus one, i.e. a complex torus. Since the topological fundamental group of a torus is $\mathbb{Z}^2$ this all makes sense.
Our final example is similarly predictable: Consider the punctured affine line $\mathbb{G}_m := \mathbb{A}^1_k \setminus \{0\}$ for $\operatorname{char}(k) = 0$. If we compare this with the topological space $\mathbb{C} \setminus \{0\}$ we would naturally expect the étale fundamental group to be $\hat{\mathbb{Z}}$ and this is indeed the case. To see this consider an étale covering $U \to \mathbb{G}_m$. No if we choose an open embedding $\mathbb{G}_m \hookrightarrow \mathbb{P}^1_k$ then we can embed U into some curve X/k such that we have a commutative diagram of k-schemes:

$$\begin{array}{ccc} U & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{G}_m & \hookrightarrow & \mathbb{P}^1_k \end{array}$$

Where the map $X \xrightarrow{f} \mathbb{P}^1_k$ is finite. In particular if we let $f^{-1}(0) = \{p_1, \dots, p_r\}$ and$f^{-1}(\infty) = \{q_1, \dots, q_s\}$ then $U = X \setminus \{p_1, \dots, p_r, q_1, \dots, q_s\}$. Now since our original map $U \to \mathbb{G}_m$ was a covering f can only be ramified at the p_i and q_i. Again denote by e_{p_i} the ramification index of f at p_i and so on. Then if we let $n = \deg(f)$ we have that $\sum_i e_{p_i} = \sum_i e_{q_i} = n$. However by Theorem 4.9 we get:

$$-2 \leq 2g(X) - 2 = -2n + \sum_{1 \leq i \leq r} (e_{p_i} - 1) + \sum_{1 \leq i \leq s} (e_{q_i} - 1) = -r - s < 0$$

And thus we have $g(X) = 0$ i.e. $X = \mathbb{P}^1_k$. Furthermore we get $r = s = 1$ and after possibly precomposing with some automorphism of $\mathbb{P}^1_k$ we may assume that f is an endomorphsim of $\mathbb{P}^1_k$ which is possibly ramified at most at the points 0 and ∞. In particular we have $U = \mathbb{G}_m$ and a commuative diagram:

$$\begin{array}{ccccc} \mathbb{G}_m & \hookrightarrow & \mathbb{A}^1_k & \hookrightarrow & \mathbb{P}^1_k \\ \downarrow & & \downarrow & & \downarrow f \\ \mathbb{G}_m & \hookrightarrow & \mathbb{A}^1_k & \hookrightarrow & \mathbb{P}^1_k \end{array}$$

The middle map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is then of degree n and thus given by some degree n polynomial $p \in k[t]$. Furthermore since $f^{-1}(0) = \{0\}$ we have that p has only 0 as a root. Consequently $p = cT^n$ for some constant c which we can remove by composing with the automorphism $T \mapsto c^{-1}T$. Note that necessarily $c \neq 0$ since our map is étale away from the point 0. Now since k is algebraically closed for each n-th root of unity ζ we get an automorphism of $\mathbb{G}_m \xrightarrow{T^n} \mathbb{G}_m$ by via multiplication with ζ. Since automatically $\# \operatorname{Aut}(\mathbb{G}_m \xrightarrow{T^n} \mathbb{G}_m) \leq n$ we have in fact equality and thus the covering is Galois. Furthermore we see that $\operatorname{Aut}(\mathbb{G}_m \xrightarrow{T^n} \mathbb{G}_m)$ is isomorphic to the group of n-th roots of unity i.e. $\mathbb{Z}/n\mathbb{Z}$. Consequently we get $\pi_1(\mathbb{G}_m) \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} \cong \hat{\mathbb{Z}}$ as claimed.

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