1 Motivation

Want to study K theory, K-groups are in general very hard to compute, e.g. $K_*(\mathbb{Z})$ is not entirely understood. It is computed in odd degrees and conjectured to vanish in degrees divisible by 4 (This is equivalent to a deep conjecture from number theory)
A toopoical Motivation for K-theory comes from the so called *Wall finiteness obstruction*, i.e. if X is a finite CW-complex and Y a retract is then Y also a finite CW-complex? As it turns out there is an obstruction to this lying in the K-group of $Z[\pi_1(X)]$.
Similiar in the famous S-cobordism Theory there is an obstruction lying in an algebraic K-theory group. This has been ugraded to a stable S-cobordism theorem by Waldhausen et.al. used for studying diffeomorphism rings.
The K-groups were originally invented by Grothendieck to give a descrtiption of the Riemann Roch theorem.

Definition 1.1.

$$K_0(R) := \{\text{ iso classes of finitely gen. projective modules over } R\}^{\text{grp}}$$

The higher K-groups are obtained as the homotopy groups of the space obtained by group completing the category of finite projective R-modules.

This homotopical maneuver is why its so complicated to compute higher K-groups. The higher K-groups of finite fields are very computable, but even for infite fields its complicated
Idea of trace methods:
Approximate by easier "more algebraic" invariants.

$$\begin{array}{ccc} K & & \\ & \searrow & \\ & TC & \longrightarrow & HC^- \\ & \downarrow & & \downarrow \\ & THH & \longrightarrow & HH \end{array} \tag{1}$$

Meta Theorem:
The map $K \xrightarrow{cyc} TC$ called the *cyclotomic trace* is often (close to) an isomorphism.
There are *relative* K groups $K_*(R, I)$ for $I \subseteq R$ an ideal defined as a cofiber. Similiarly for $TC(R, I)$. Also have versions wi *with coefficients* $K(R, \mathbb{Z}_p), TC(R, \mathbb{Z}_p)$

Theorem 1.2. – *If $I \subseteq R$ is a nilpotent ideal then the relative trace*

$$K_*(R, I) \to TC_*(R, I)$$

is an ismorphism. Conceptually the difference between K-theory and TC is nilinvariant.

– *(Clause Matthew Morrow) If R is commutative and I-complete then the map $K(R, I, \mathbb{Z}_p) \to TC(R, I, Z_p)$. This can be used to compute the K-theory of p-adic rings*

– *If R is p-complete then*

$$TC(R, \mathbb{Z}_p) \cong K^{ét}(R, Z_p)$$

Where the right hand side is the étale sheafification of K-theory.

Let k be field and R a k-algebra, P a finitely generated right module over R.

$$\mathrm{Hom}_R(P, P) \cong P \otimes_R \mathrm{Hom}_R(P, R)$$

There is no well defined trace map since the naive evaluation gives:

$$x \otimes r\phi \mapsto r\phi(x)xr \otimes \phi \qquad \mapsto \phi(xr) = \phi(x)r$$

Take the abelianization $R^{\text{ab}} = R/[R,R]$, then we get a trace map:

$$\text{Hom}_R(P,P) \to R^{\text{ab}}$$

Recall that a left R-module is equivalently a right R^{op} module. An $R-R$-bimodule is an $R \otimes_k R^{\text{op}}$-module. In this language:

$$R^{\text{ab}} \cong R \otimes_{R \otimes_k R^{\text{op}}} R$$

And in fact we will see that

$$HH(R/k) \cong R \otimes^L_{R \otimes_k R^{\text{op}}} R$$

furthermore there exists a map called the *Dennis trace*

$$\text{tr} : K(R) \to HH(R/k)$$

Which refines the construction we gave earlier in the sense that on π_0 its given by the actual trace of the identity $P \xrightarrow{\text{id}} P$ we constructed above.

2 Classical Hochschild Homology

2.1 Definitions

Let k be a field and R a k-algebra.

Definition 2.1. Define $HH(R/k)$ as the homology of the complex associated to the cyclic bar resollution

Example: $HH(k/k) = k$ concentrated in degree zero.
Now assume that R is commutative, the we immediately see that $HH_o(R/k) = R$ and $HH_1(R/k) = \frac{R \otimes_k R}{\text{im}(d)} \cong \Omega^1_{R/k}$ on the nose by the usual description as I/I^2 where I is the kernel of the multiplication map. Now want to give a discription of the higher groups:

Lemma 2.2. *R commutative, then $HH_*(R)$ has the structure of a strictly graded commutative ring. [Strictly means that $x^2 = 0$ if $|x|$ is odd]*

Proof. Follows from the fact that the hochschild complex $HH_*(R/k)$ is associated to an animated ring by definition. ☐

Definition 2.3. Define $\Omega^*_{R/k} := \bigwedge^*_R \Omega^1_{R/k} = \frac{\text{Sym}\Omega^1_{R/k}[1]}{x^2=0 \ | \ x \in \Omega^1_{R/k}}$

Hence we get a map $\Omega^*_{R/k} \to HH_*(R/k)$ of graded commutative rings. [It does not take into account the differential on the left hand side]

Theorem 2.4. *If R/k has cotangent complex concentrated in degree 0 then this map is an iso. Hence in particular for R and (ind)-smooth k algebra*

Excercise: Find an explicit description of the DeRahm complex for a Polynomial ring in n variables and show that it vanishes in degrees larger than n.
Now let k be an arbitrary commutative ring. This leads to issues since not everything is flat. For now follow this ad-hoc approach:
R a differential graded algebra over k which is K-flat. [This is a subtle condition but in good cases it agrees with levelwise flatness]. For such a dga define:

$$HH(R/k) = H(\text{Tot}\,\text{B}^{\text{cyc}}(R))$$

Where the cyclic bar resolution is now a bicomplex hence we take the Homology of the total complex. If R is not K-flat we can find a K-flat replacement $R^\flat$ and define:

$$HH(R/k) := HH(R^\flat/k)$$

We will see that this is well defined upto quasi isomorphism.

Proposition 2.5. *For a dga R have that:*

$$HH(R/k) \cong R \otimes^L_{R\otimes^L_k R^{\mathrm{op}}} R$$

Proof. Replace R by a flat resolution. Consider the regular bar complex of R:

$$\cdots \to R \otimes_k R \otimes_k R \to R \otimes_k \simeq R$$

Note that term is a K-flat $R \otimes_k R^{\mathrm{op}}$-module. Applying $- \otimes_{R\otimes_k R^{\mathrm{op}}} R$ yields the cyclic bar complex. Hnece it follows from the usual description of derived tensor products that the total complex computes both the left and the right hand side. as above. □

Lemma 2.6. *If R is a dga which arises from an animated k-algebra, then so does $HH(R/k)$. In particular $HH_*(R/k)$ form a strictly graded-commutative k-algebra.*

Proof. We can replace R by a level wise flat animated k-algebra. Then we can apply Hochschild homology levelwise $HH(R_n)$ gives a *bisimplicial* commutative k-algebra. Then the total complex is quasi ismorphic to the the complex associated to the diagonal. □

Lemma 2.7. *A, B dga's, then:*

$$HH(A \otimes^L_k B/k) \simeq HH(A/k) \otimes^L_k HH(B/k)$$

Proof. Idea: Replace A, B by resolving with K-flat dga's. Then $HH(A/k), HH(B/k)$ are simplicial dga's so we can regard $HH(A/k) \otimes_k HH(B/k)$ is a bisimplicial dga. The total complex is however again quasi isomorphic to the total complex of the diagonal which is by definition $HH(A \otimes_k B)$ □

We are now ready to prove the HKR theorem for polynomial algebras.

Proof. (For polynomial rings over fields)

- $R = k[x]$ then $\Omega(R/k) = k[x] \otimes \bigwedge(dx)$
  Since this is concentrated in degrees 1 and 2 the theorem only asserts that the Hochschild Homology vanishes in higher degrees. Will use:

  $$HH(k[x]/k) = k[x] \otimes^L_{k[x]\otimes_k k[x]^{\mathrm{op}}} k[x] \cong k[x] \otimes^L_{k[a,b]} k[x]$$

  Need to resolve the terms over $k[a, b]$, can do this as:

  $$k[a, b] \xrightarrow{\cdot(a-b)} k[a, b]$$

  Thus the Tor groups vanish in higher degrees as desired

- $R = k[x_1, \cdots, x_n]$ this is an n-fold tensor product of the first case. But both sides commute with tensor products

- if R is polynomial in an arbitrary number of generators. It is then a filtered colimit of the second case and the claim follows that both sides commute with filtered colimits. This is the case since the DeRahm complex admits a generators + relations description and HH is built from derived tensor products.

$\mathrm{Hom}^{ex}_{\mathcal{C}}(X, Y)$ □

Proposition 2.8. *Have that $HH_*(\mathbb{F}_p/\mathbb{Z}) = \frac{\mathbb{F}_p[x_1, x_2, \cdots]}{x_i x_j = \binom{i+j}{i} x_{i+j}} = \mathbb{F}_p < x >$ " $\mathbb{F}_p\{1, x, \frac{x^2}{2!}, \cdots\}$"*

Proof. compute $\mathbb{F}_p \otimes^L_{\mathbb{F}_p \otimes^L_{\mathbb{Z}} \mathbb{F}_p^{\text{op}}} \mathbb{F}_p$. We can replace as as:

$$\mathbb{F}_p \simeq \mathbb{Z}[\varepsilon]/\varepsilon^2, \quad \partial\varepsilon = p \;.\; |\varepsilon| = 1$$

Then get that:

$$A \simeq \mathbb{F}_p \otimes^L_{\mathbb{Z}} \mathbb{F}_p^{\text{op}} \simeq \mathbb{F}_p[\varepsilon]/\varepsilon^2 \quad \partial\varepsilon = 0$$

Consider $A\langle x\rangle$ with differential $\partial x_i = \varepsilon x_{i-1}$, $\partial x_1 = \varepsilon$ and $|x_i| = 2i$. This defines a flat resolution of $\mathbb{F}_p$.
Exercise : Check that this defines a dga (i.e. that d is a derivation) and that the homology is $\mathbb{F}_p$ concentrated in degree zero
Hence get :

$$A\langle x\rangle \otimes_A \mathbb{F}_p = \mathbb{F}_p\langle x\rangle$$

as desired. □

Remark 2.9. The Eilenberg-Zilber Theorem says precisely that the Dold Kan correspondence is lax symmetric monoidal.

2.2 The Connes Operator on Hochschild Homology

The Connes Operator is degree 1 self map of HH which roughly corresponds to the deRahm differential under the HKR theorem. Setup: k commutative ground ring, R a k-algebra

$$HH_*(R/k) = H_*(HH(R/k))$$

where

$$HH(R/k) = (\cdots \to R \otimes^L_k R \xrightarrow{\partial} R) \in D(k)$$

Theorem 2.10 (HKR). *If $L_{R/k}$ has flat dimension 0, then we have an isomorphism of graded commutative rings:*

$$HH_*(R/k) \cong \Omega^*_{R/k}$$

Where on the left hand side we really mean the Homology of the Hochschilod complex,so his isomorphism as a priori nothing to do with chain complex structure on the right hand side!

Definition 2.11. Let R be an associative k-algebra. We define a k-linear map:

$$B : HH(R/K)_n \to HH(R/k)_{n+1}$$
$$r_o \otimes \cdots \otimes r_n \mapsto \sum_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)}(1 \otimes r_\sigma - (-1)^n r_\sigma \otimes 1)$$

Where $r_\sigma = r_{\sigma(0)\otimes r_{\sigma(n)}}$ and we think of C_{n+1} as being the group of cyclic permutations of $\{0, \ldots, n\}$.

Proposition 2.12 (Exercise). *Check that:*

1. $B^2 = 0$
2. $\partial B + B\partial = 0$

The Operator B equips $HH(R/k)$ with the structure of a differential graded module over the DGA:

$$A = \frac{\mathbb{Z}[b]}{b^2}, \quad |b| = 1, \partial = 0$$
$$= H_*(\Pi, \mathbb{Z}), \quad \Pi = U(1)$$

where the product structure on the last term is the Pontryagin product.
This implies that $HH_*(R/k)$ is a graded module over $\frac{Z[b]}{b^2}$. In particular there is an operator:

$$B : HH_*(R/k) \to HH_{*+1}(R/k) \quad \text{with } B^2 = 0$$

Note that we have not technically defined the operator B for a dga, but the exact same definition works there as well and everything else goes through as well.

Proposition 2.13. *If R is commutative, then B is a derivation, i.e. it satisfies the graded Leibniz rule:*

$$B(xy) = B(x)y + (-1)^{|x|}xB(y)$$

Warning: This is not true on the nose on the complex $HH(R/k)$ but only up to coherent chain homotopy. We will prove this later in a more systematic way.

Proposition 2.14. *The map*

$$\Omega^*_{R/k} \to HH_*(R/K)$$

sends the deRham differential d to the Connes operator B, i.e. the following square commutes:

$$\begin{array}{ccc} \Omega^*_{R/k} & \longrightarrow & HH_*(R/k) \\ \downarrow d & & \downarrow B \\ \Omega^{*+1}_{R/k} & \longrightarrow & HH_{*+1}(R/k) \end{array}$$

Proof. The map $\Omega^*_{R/k} \to HH_*(R/k)$ is determined bu its effect in degees $0, 1$:

deg 0: $R \to HH_o(R/k) = R$

deg 1: $\Omega^1_{R/k} \to HH_1(R/k) \simeq \frac{R \otimes_k M}{\sim}$
$x\mathrm{d}y \mapsto [x \otimes y]$

Thus in order to show the statement it is enough to check that:

$$\begin{array}{ccc} R = \Omega^0_{R/k} & \longrightarrow & HH_0(R/k) \\ \downarrow d & & \downarrow B \\ \Omega^1_{R/k} & \longrightarrow & HH_1(R/k) \end{array} \qquad \begin{array}{ccc} r & \longmapsto & [r] \\ \downarrow & & \downarrow \\ dr & \longleftarrow & [1 \otimes r - r \otimes 1] \end{array}$$

□

We will abuse this result by referring to the B operator as d. To avoid confusion we try to stick to the convention of referring to the differential on the Hochschild chain complex as ∂.

Exercise 2.15. *For $HH_*(\mathbb{F}_p/Z) \simeq \mathbb{F}_p\langle x\rangle$ with $|x| = 2$ the Connes operator acts trivially for degree reasons.*
Question: Does this mean that it is "trivial" on $HH(\mathbb{F}_p/\mathbb{Z})$?

Consider the (derivedf) mod p reduction:

$$\begin{aligned} HH(\mathbb{F}_p)//p &:= HH(\mathbb{F}_p) \otimes^L_{\mathbb{Z}} \mathbb{F}_p \\ &\simeq HH(\mathbb{F}_p) \otimes_{\mathbb{Z}} (\bigwedge_{\mathbb{Z}}(\varepsilon), |\varepsilon| = 1, \partial\varepsilon = p) \\ &\simeq HH(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \bigwedge_{\mathbb{F}_p}(\varepsilon) \end{aligned}$$

Thus on Homology we get:

$$\begin{aligned} H_*(HH(\mathbb{F}_p)//p,) &\cong HH_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \bigwedge_{\mathbb{F}_p}(\varepsilon) \\ &\cong \mathbb{F}_p\langle x\rangle \otimes_{\mathbb{F}_p} \bigwedge_{\mathbb{F}_p}(\varepsilon) \end{aligned}$$

Proposition 2.16. *We have that:*

$$B(x^{[n]}) = 0, \quad B(\varepsilon) = x$$
$$\left(\implies B(\varepsilon x^{[n]}) = x x^{[n]} = (n+1)x^{[n+1]} \right)$$

Proof. Note that the map $HH(\mathbb{F}_p) \to HH(\mathbb{F}_p)//p$ is compatible with B. Thus we get the first claim $B(x^{[n]}) = 0$ on Homology.
The way $HH_*(\mathbb{F}_p) = \mathbb{F}_p\langle x \rangle$ is computed is by replacing $\mathbb{F}_p$ by $\bigwedge_{\mathbb{Z}}(\varepsilon), \partial\varepsilon = p$. The cyclic Bar complex has the form:

$$\dots \xrightarrow{\partial} \bigwedge_{\mathbb{Z}}(\varepsilon) \otimes \bigwedge_{\mathbb{Z}}(\varepsilon) \xrightarrow{\partial} \bigwedge_{\mathbb{Z}}(\varepsilon)$$

Written as double complex:

$$\begin{array}{ccccccc}
 & & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \dots & \longrightarrow & \mathbb{Z}(\varepsilon \otimes \varepsilon) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (-p,p)} & & \downarrow \\
 & & \dots & \longrightarrow & \mathbb{Z}(\varepsilon \otimes 1) \oplus \mathbb{Z}(1 \otimes \varepsilon) & \xrightarrow{0} & \mathbb{Z}\varepsilon \\
 & & \downarrow & & \downarrow{\scriptstyle (p,p)} & & \downarrow \\
\dots & \longrightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}
\end{array}$$

Here the class x is given by $1 \otimes \varepsilon - \varepsilon \otimes 1$. We have that $B(\varepsilon) = 1 \otimes \varepsilon - \varepsilon \otimes 1$. After mod p reduction we replace all $\mathbb{Z}$ by $\mathbb{F}_p$ then ε becomes a cycle and $B([\varepsilon]) = [x]$ □

3 Periodic and cyclic Homology

In the following R a k-algebra, $HH(R/k)$ the Hochschild complex and $B : HH(R/k) \to HH(R/k)[-1]$, another way to encode this is saying that $HH(R/k)$ is a DG-module over DGA:

$$A = \left(\frac{k[b]}{b^2}, |b| = 1, \partial = 0 \right)$$

Write $\mathrm{DGMod_A} \to \mathrm{DGMod}_A[\text{ quasi-isos }^{-1}] \simeq \mathrm{Mod}_A(Dk)$ meaning where using the DGAs as a model for the categories of modules over the derived category of k.

Definition 3.1. 1. The *cyclic Homology* of R is

$$HC_*(R/k) := \left(k \otimes_A^L HH(R/k)\right) = \mathrm{Tor}_*^A(k, HH(R/k))$$

2. The *negative cyclic Homology* of R is:

$$HC_*^-(R/k) = H_*(\mathrm{RHom}_A(k, HH(R/k))) = \mathrm{Ext}_A^{-*}(k, HH(R/k))$$

This is a module over $\mathrm{Ext}_A^{-*}(k,k) \simeq k[t]$ with $|t| = 2$

3. The *Periodic Homology* of R is:

$$HP_*(R/k) = HC_*^-(R/k)[t^{-1}] \simeq \underrightarrow{\mathrm{colim}} \left(HC_*^- \xrightarrow{\cdot t} HC_*^- \xrightarrow{\cdot t} \dots \right)$$

Similar to our notation for Hochschild Homology if we omit the $*$ index we mean the complexes/homotopy types.

These are not the definitions one would find usually in the literature, the connection is the following:

Proposition 3.2. *For any k-algebra R we have that:*

$$HC^-(R/k) \simeq (HH(R)[|t|]), |t| = 2, \partial + tB)$$

where $\partial = tB$ is defined as:

$$xt^n p \mapsto t'\partial x + (Bx)t^{n+1}, \qquad \text{for } x \in HH(R/k)$$

and similarly:

$$HP(R/k) \simeq (HH(R)((t)), \partial + tB)$$

Proof. Have to resolve k as an$A$-algebra by:

$$C = A\{x_0, x_1, \dots\}, \quad |x_k| = 2k$$
$$\partial(x_k) = bx_k - 1$$

In fact C is a coalgebra with $\Delta(x_k) = \sum_{i+j=k} x_i \otimes x_j$.

$$\begin{aligned} \mathrm{RHom}_A(k, HH(R/k)) &= \underline{\mathrm{Hom}}_A(C, HH(R/k)) \\ &\simeq (HH(R)[|t|], \partial + tB) \end{aligned}$$

The formula for HP follows by localizing at t. □

Remark 3.3. – This formula works more generally for any object $H \in \mathrm{DgMod_A}$ to give $\mathrm{RHom}(k, H)$.

– A is a Hopf algebra via:

$$\varepsilon : A \to k \ , \ \varepsilon(b) = 0$$
$$\Delta : A \to A \otimes A \ , \ \Delta(b) = 1 \otimes b + b \otimes 1$$

Claim: This induces a symmetric monoidal structure on the category of differential graded A-modules such that k is the tensor unit.[How? This is confusing me] This implies that $\underline{\mathrm{RHom}}(k, -)$ is a lax symmetric monoidal functor and as such it is given by $H \mapsto (H[|t|], \partial + tB)$. So in particular if C is an algebra object in DgMod_A, i.e. C is a DGA and B satisfies the Leibniz rule, then $\underline{\mathrm{RHom}}(k, C)$ is a DGA.

Problem: Even if R is commutative, B is not on the nose a derivation on $HH(R/k)$, but it works up to chain homotopy. However this does not suffice to induce an algebra structure on the derived Hom, we need it up to coherent homotopy, which does hold but we do not have the machinery to talk about it yet. But there is a Hack: It does suffice to dedeuce that $(HH(R/k)[|t|]), \partial + B)$ has a product up to chain homotopy upt o homotopy, i.e. that $HC_*(R/k)$ and $HP_*(R/k)$ are graded commutative algebras.

Definition 3.4. Let R be a commutative k-algebra. The *De Rham Cohomology* of R relative to k is defined as:

$$H^*_{dR}(R/k) = H_*(\Omega^*_{R/k}, d)$$

Remark 3.5. H^*_{dR} can (and should) be defined for shcemes X/k, it is often boring in the affine case.

Theorem 3.6 (A)**.** *Assume that $\mathbb{Q} \subset k$ and $L_{R/k}$ is flat of dimension 0. (e.g. R smooth). Then there are antural isomorphisms:*

$$HP_*(R/k) \cong H^*_{dR}(R/k)((r)) \quad |t| = -2$$
$$HC^-_n = Z^n_{dR}(R/k) \oplus \prod_{i\geq 1} H^{n+2i}_{dR}(R/k)$$

Where Z_{dR} denote the cycles in the deRahm complex.

Proposition 3.7 (Exercise)**.** *Describe the map $HC^- \to HP$ in this case and show, that it exhibits HP as the localization a t. [I am not sure i understand the grading on the first one]. The map sends the first summand to the quotient and a sequence of element $(x_1, x_2, \dots)$ to the power series $\sum_{i\geq 1} x_i t^i$.*

Remark 3.8. – this statement as written is often trivial, since both sides tend to vanish on affine space, it is supposed to model the real statement. In fact for a scheme X/k with $L_{X/R}$ flat of dimension 0 we have:

$$HP_*(X/k) \cong H^*_{dR}(X/k)((t))$$

– These statements are false if $\operatorname{char} k$ is not 0.

Consider the map

$$\mu : R^{\otimes(n+1)} HH(R/k)_n \to \Omega^n_{R/k}$$
$$r_0 \otimes \cdots \otimes r_n \mapsto \frac{1}{n!} r_o \mathrm{d}r_1 \cdots \mathrm{d}r_n$$

Lemma 3.9. *This is a map of CDGA's:*

$$(HH(R/k), \partial, B) \to (\Omega^*_{R/k}, 0, d)$$

proof of the lemma.

$$\begin{aligned}
\mu(\partial(r_o \otimes \cdots \otimes r_n) &= \mu\left(r_0 r_1 \otimes r_2 \otimes \cdots \otimes r_n - r_1 \otimes r_2 r_3 \otimes \cdots \otimes r_n + \cdots \pm r_o r_1 \otimes r_1 \otimes \cdots \otimes r_{n-1})\right) \\
&= r_0 r_1 \mathrm{d}r_2 \cdots \mathrm{d}r_n - r_0 \mathrm{d}(r_1 r_2) \mathrm{d}r_3 \dots \mathrm{d}_n \\
&= r_0 r_1 \mathrm{d}r_2 \cdots \mathrm{d}r_n - r_0 r_1 dr_2 dr_3 \cdots [....] = 0
\end{aligned}$$

Complicated to write out, all the terms cancel.

Exercise 3.10 (Exercise)**.** *Check that μ is mulitplicative with respect to the shuffle product on the left and wedge product of forms on the right.*

□

Corollary 3.11. *If R has flat $L_{R/k}$ and k is rational we have:*

$$(HH(R/k), \partial, B) \simeq (\Omega^*_{R/k}, 0, d)$$

Proof. We always have that the composition

$$\Omega^*_{R/k} \to HH_*(R/k) \xrightarrow{H_*(\mu)} \Omega^*_{R/k} \tag{2}$$

is the identity, so the statement follows from the HKR theorem. □

For Theorem A we get:

$$\begin{aligned} HP_*(R/K) &= H_*HH((R/k)((t)), \partial + B) \\ &\simeq H_*(\Omega^*_{R/k}((t)), td) \\ &\simeq H^*_{dR}(R/k)((t)) \end{aligned}$$

And similarly for $HC^*_*(R/k) \simeq H_*(\Omega^*[|t|], td)$. This proves the theorem.
All of this crucially relies on us working in characteristic 0 as we will see.

<u>Construction</u> If k is arbitrary we have the Postinkov filtration

$$\tau_{\geq \bullet} HH(R/k)$$

on $HH(R/k)$ which is compatible with all the structure and leads to a filtration on $HP(R/k)$ concretely given by:

$$\tau_{\geq \bullet}(HH(R/k)((t)),\ \partial + tB)$$

and leadss to a multiplicative conditionally convergent spectral sequence

$$E_2 = HH_*(R/k)((t)) \Rightarrow HP_*(R/k)$$

since the Hochschild complex is the associated graded of the latter filtration. The E_3-page is given by:

$$H_*(HH_*(R/k), B)((t)) \Rightarrow HP_*(R/k)$$

If R has flat cotangent complex concentrated in one degree then this is:

$$E_3 = H^*_{dR}(R/k)((t)) \Rightarrow HP_*(R/k)$$

Definition 3.12. R commutative ring, we define the *divided power series algebra* $R\langle\langle x\rangle\rangle$ as the completion of $R\langle x\rangle$ at the filtration given by the divided powers of x. Note that this is *not* the adic fitration given by x.

Proposition 3.13. *For $A = \mathbb{F}_p$ we hvae that :*

$$HC^-_*(\mathbb{F}_p/Z) \simeq \frac{\mathbb{Z}_p[t]\langle\langle x\rangle\rangle}{xt - p} \qquad |x| = 2\ ,\ |t| = 2$$

and:

$$HP_*(\mathbb{F}_p/\mathbb{Z}) \simeq \frac{\mathbb{Z}_p[t^{\pm}\langle\langle x\rangle\rangle}{xt - p} = \left(\frac{\mathbb{Z}_p\langle\langle y\rangle\rangle}{y - p}[t^{\pm}]\right) \qquad |y| = 0$$

Remark 3.14.

- The ring:

  $$HP_o(\mathbb{F}_p/\mathbb{Z}) \cong HC^-_o(\mathbb{F}_P/Z) \cong \frac{\mathbb{Z}\langle\langle y\rangle\rangle}{y - p}$$

  Is obtained by adjoining divided powers of p to $\mathbb{Z}_p$. This is a weird ring since $\mathbb{Z}_p$ already has divided powers of p. In fact this process gives us p-torison since $y^{[p]} - \frac{p^p}{p!}$ is $p!$ and hence p-torsion
- In fact we have that $\frac{\mathbb{Z}_p[y]}{y-p} \cong \frac{\mathbb{Z}_p\langle z\rangle}{z}$ which is <u>not</u> $\mathbb{Z}_p$!
- In fact $HP_*(\mathbb{F}_p/Z)$ is 2-periodic derived deRahm cohomology of $\mathbb{F}_p$ relative $\mathbb{Z}$.

Proof. Recall that $HH_*(\mathbb{F}_p/Z) \cong \mathbb{F}_p\langle x\rangle$ and the spectral sequence we gave earlier:

$$\mathbb{F}_p\langle x\rangle[t] \Rightarrow HC^-(\mathbb{F}_p/Z)$$

which immediately collapses because everything is evenly graded. Thus $HC^-_*(\mathbb{F}_p/Z)$ has a associated graded given by $\mathbb{F}_p\langle x\rangle[t]$. Need some facts:

- The connective cover of $HC^-(\mathbb{F}_p/\mathbb{Z})$ can be represented by a simplicial commutative ring. Thus it admits divided powers on positive degree homotopy groups (will discuss this later). In particular every choice of $x, t \in H_*^-(\mathbb{F}_p/\mathbb{Z})$ gives a map:

$$\mathbb{Z}\langle x\rangle[t] \to HC_*^-(\mathbb{F}_p/Z)$$

if we can find x, t such that $xt = p$ in $HC_*^-(\mathbb{F}_p/\mathbb{Z})$ then this map induces an isomorphism on associated gradeds of the map:

$$\frac{\mathbb{Z}\langle x\rangle}{xt - p} \to HC_*^-(\mathbb{F}_p/\mathbb{Z})$$

Back to the computation of $HH(\mathbb{F}_p/\mathbb{Z})$, there we computed:

$$\mathbb{F}_p \cong \left(\bigwedge_{\mathbb{Z}}(\varepsilon), \partial\varepsilon = p\right)$$

$$x = B\varepsilon \ , \ partial x = 0 \ , \ Bx = 0$$

From this we get that in $HH(\mathbb{F}_p/\mathbb{Z})$ x represents a cycle $x \in HH_*(\mathbb{F}_p/\mathbb{Z})$ and hence for :

$$HC^-(HH(\mathbb{F}_p/\mathbb{Z}[|t|], \partial + tB)$$

we get:

$$(\partial + tB)\varepsilon = p + tx \implies p = tx \in HC_*^-(\mathbb{F}_p/\mathbb{Z})$$

□

Remark 3.15. – One can also deduce this computation using that:

$$HH(\mathbb{F}_p/\mathbb{Z}, \partial, B) \simeq (\frac{\mathbb{Z}[\varepsilon]}{\varepsilon^2}\langle x\rangle \ , \ \partial\varepsilon = p \ , \ \partial x = 0 \ , \ B\varepsilon = x \ , \ Bx = 0)$$

- Once can construct long exact sequences:

$$\cdots \to HC_{*-1}(R) \to HC_*^-(R/) \to HP_*(R/k) \to \ldots$$

This is quite useful but we do not want to talk about HC, which actually does not have a ring structure. This sequence tells us, that we can think of it as an ideal/kernel of the map $HC^- \to HP$

4 derived functors

Recall: For $\mathcal{A}$ an abelian category have the derived (∞)-category $D(\mathcal{A}) = K(A)[W]^{-1}$ where W was the class of quasi isos.
Now let $F\mathcal{A} \to B$ be an additive functor i.e. it preserves the zero object and direct sums.
Construction:

From F we get an induced functor:

$$Ch(\mathcal{A}) \xrightarrow{Ch(F)} Ch(\mathcal{B})$$

by applying F level wise. This is well defined since by additivity F maps the zero map to the zero map. This in fact refines to an enriched functor over $Ch(\mathbb{Z})$. Hence we get a functor

$$K(\mathcal{A}) = N_{dg}(Ch(\mathcal{A}) \xrightarrow{N_{dg}(F)} N_{dg}(Ch(\mathcal{B})) = K(\mathcal{B})$$

which commutes with finite limits and colimits i.e. it is exact. This can be checked immediately from the explicit description of pullbacks. We would like to get an induced functor $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$ such that we get a commutative square:

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}) \\ \downarrow{\scriptstyle pr_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{pr}_{\mathcal{B}}} \\ D(\mathcal{A}) & \xrightarrow{D(F)} & D(\mathcal{B}) \end{array}$$

of infinity categories. One might be inclined to use the universal property of the DK-localization to define such a map. However we have the following:

Proposition 4.1. *In this setting the following are equivalent:*

1. *There exists a functor $D(\mathcal{A}) \to D(\mathcal{B})$ making the square commute*
2. *The functor $K(\mathcal{A}) \to K(\mathcal{B})$ preserves quasi-isomorphisms*
3. *The functor $F : \mathcal{A} \to \mathcal{B}$ is exact*

Proof. Indeed: the equivalence $1 \iff 2$ is clear from the universal property and $2 \iff 3$ is a standard fact from homological algebra (since quasi isos are detected on the level of homotopy categories). $\square$

But of course we are interested in non-exact functor, hence we cannot hope for such a commutative square as such.

Definition 4.2. The *left derived functor LF* of F is given by a functor $LF : D(\mathcal{A} \to \mathcal{D})$ together with a natural transformation

$$\eta : LF \circ p_{\mathcal{A}} \to p_{\mathcal{B}} \circ K(F)$$

such that for any other functor $H : D(\mathcal{A} \to D(\mathcal{B}))$ such that the induced map:

$$\mathrm{Map}_{\mathrm{Fun}(D(\mathcal{A}),D(\mathcal{B}))}(H, LF) \to \mathrm{Map}_{\mathrm{Fun}(K(\mathcal{A}),K(\mathcal{B}))}(H \circ p_{\mathcal{A}}, p_{\mathcal{B}} \circ K(F))$$

is an equivalence.

Warning: The left derived functor might not exist in our generality (This actually happns)

Proposition 4.3. *If LF exists it preserves finite limits and colimits. [Check!]*

Definition 4.4. Given an $(\mathcal{C}, W)$ and a functor $G : \mathcal{C} \to \mathcal{D}$ then the left derived functor LG is given by:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & \overset{\eta}{\nearrow\!\!\!\nearrow} & \nearrow{\scriptstyle LG} \\ \mathcal{C}[W^{-1}] & & \end{array}$$

such that the induced map

$$\mathrm{Map}(H, LG) \xrightarrow{\eta^*} \mathrm{Map}(H, G)$$

is an equivalence for each $H : \mathcal{C}[W^{-1}] \to \mathcal{D}$

We say that LG is the *absolute* left derived functor if for any

$$T : \mathcal{D} \to \mathcal{E}$$

the induced triangle:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} & \xrightarrow{T} & \mathcal{E} \\ \downarrow & \overset{T\circ\eta}{\nearrow\!\!\!\nearrow} & & \nearrow{\scriptstyle T\circ LG} & \\ \mathcal{C}[W^-1] & & & & \end{array}$$

exhibits $T \circ LG$ as the left derived functor of $T \circ G$.

Remark 4.5. We have that

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \cong \mathrm{Fun}^W(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where Fun^W denotes the full subcategory of functor which send weak equivalences to equivalences. For given $G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ we have that $LG \in \mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$ hence LG is a *coreflection* of G.

Theorem 4.6. *For any functor $G : K(\mathcal{A}) \to \mathcal{D}$ we have that:*

$$LG(C) = \varprojlim_{\hat{C} \xrightarrow{\text{q.iso}} C} G(\hat{C})$$

provided that this limit exists for each $C \in K(\mathcal{A})$ (This is again the limit over the slice ${K(\mathcal{A})}/_{C}^{\text{q.iso}}$)

Exercise 4.7. *1. For $C = 0$ we have that:*

$$(LG)(0) = \varprojlim_{\hat{C} \to C} G(\hat{C}) = G()$$

since 0 is inital.

2. For any C assume that there is a K-projective object with a quasi iso $p : \hat{C} \xrightarrow{\sim} C$ then we claim that p is inital in $K(\mathcal{A})^{\text{q.iso}}/C$. To see that we note that for any other object $C' \xrightarrow{\sim} C$ we have a pullback:

$$\begin{array}{ccc} \mathrm{Map}_{K)\mathcal{A}/C}(\hat{C} \to C, \hat{C} \to C') & \longrightarrow & \mathrm{Map}_{K(\mathcal{A})}(\hat{C}, C') \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Delta^0 & \xrightarrow[p]{\qquad\qquad} & \mathrm{Map}_{K(\mathcal{A})}(\hat{C}, C) \end{array}$$

Where the right vertical map is an equivalence since $\hat{C}$ was K-projective and hence since pullbacks preserves equivalences so is the left vertical map.
So we see that:

$$(LG)(C) = H(\hat{C})$$

i.e. we can compute the left derived functor via projective resolutions.

Note: If for any object C there exists sucha K-projective resolution $\hat{C}$ then

$$LG(C) = G(\hat{C})$$

for any functor G and in particular this is compatible with postcomposition with functos $T : \mathcal{D} \to \mathcal{E}$ i.e. left derived functors exist and they are always absolute.
Example: Assume that $F : \mathcal{A} \to clB$ additive, $\mathcal{A}$ is small and $\mathcal{B}$ has all infinite products and they are exact. Then we have that:

- $D(\mathcal{B})$ has all limits (Check!)
- The derived functor $LF : D(A) \to D(B)$ exists
- LF preserves finite colimits and limits.

There is a obvious dual notion of right derived functors. Then the dual of our formula (using that $K(\mathcal{A}^{op}) = K(\mathcal{A})^{op}$) read as:

$$Rg(C) = \varinjlim_{C \xrightarrow{\sim} \hat{[C]}} G(\hat{C})$$

Example: We have functors:

$$\mathrm{Map}_{K(\mathcal{A})}(A,-) : K(\mathcal{A}) \to \mathcal{S}$$

$$\mathrm{Map}_{K(\mathcal{A})}(-,B) K(A)^{\mathrm{op}} \to \mathcal{S}$$

Then the formulas for $\mathrm{Map}(A,B)$ show that:

$$\mathrm{Map}_{D(\mathcal{A})} \simeq \mathrm{RMap}_{K(\mathcal{A})}(A,-)$$

$$\mathrm{Map}_{D(\mathcal{A})}(-,B) \simeq \mathrm{RMap}_{K(\mathcal{A})}(-,B)$$

We now sketch the proof of the formula for the left derived functor:

Proof. Want to show that:

$$LG(C) \simeq \varprojlim_{\hat{C} \xrightarrow{\sim} C} G(\hat{C})$$

if this limits exist.

1. One shows that the formula above defines a functor $K(\mathcal{A}) \to \mathcal{D}$
2. It preserves weak equivalences. To prove this one considers the pullback functor between slice categories which turn out to have an adjoint.
3. There is a natural transformation $\eta : LG \to G$ which is immediate from (1)
4. If G preserves quasi isos then η is an equivalence
5. For any H which preserves quasi isomorphisms check that the map:

   $$\mathrm{Map}(H, LG) \xrightarrow{\eta_*} \mathrm{Map}(H,G) \xrightarrow{L} \mathrm{Map}(LH, LG) \simeq \mathrm{Map(H,LG)}$$

   is the identity (+ the other composition)

□

Definition 4.8. Let $\mathcal{D}, \mathcal{D}$ be ∞-categories with functors:

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

Then a natural transformation $\varepsilon : LR \to \mathrm{id}_{\mathcal{D}}$ is called the *counit of an adjunction* if for a pair of objects $c \in \mathcal{C}, d \in \mathcal{D}$ the induced map:

$$\mathrm{Map}_{\mathcal{C}}(c, Rd) \xrightarrow{L} \mathrm{Map}_{\mathcal{D}}(Lc, LRd) \xrightarrow{\varepsilon_d} \mathrm{Map}_{\mathcal{D}}(Lc, d)$$

is an equivalence. Dually a transformation $\eta : \mathrm{id}_{\mathcal{C}} \to RL$ is called *unit of an adjunction* if the induced map:

$$\mathrm{Map}_{\mathcal{D}}(Lc, d) \xrightarrow{R} \mathrm{Map}_{\mathcal{C}}(RLC, Rd) \xrightarrow{\eta_c^{pt}} \mathrm{Map}_{\mathcal{C}}(c, Rd)$$

is an equivalence. In either case we say that L is left adjoint to R and write $L \dashv R$

Given L, R as above and $\varepsilon : LR \to \mathrm{id}, \eta : \mathrm{id} \to RL$ any pair of natural transformation we say that the *Zig Zag identities* hold if the composites:

$$L = L \circ \mathrm{id} \xrightarrow{\eta} L \circ R \circ L \xrightarrow{\varepsilon} \mathrm{id} \circ L = L$$

$$R = \mathrm{id} \circ R \xrightarrow{\eta} R \circ L \circ R \xrightarrow{\varepsilon} R \circ \mathrm{id} = R$$

are equivalent to id_L respectively id_R.

Proposition 4.9. *If transformations ε and η satisfy the ZigZag identities then these are unit and counit of an adjunction. Conversely given a unit η of an adjunction then there is a unique counit ε such that the ZigZag identities are satisfied and vice versa.*

Proof. Assume that ε and η satisfy the ZigZag identities then:

$$\mathrm{Map}_{\mathcal{C}}(c, Rd) \xrightarrow{L} \mathrm{Map}_{\mathcal{D}}(Lc, RLd) \xrightarrow{\varepsilon} \mathrm{Map}_{\mathcal{D}}(Lc, d)$$

$$\mathrm{Map}_{\mathcal{C}}(Lc, d) \xrightarrow{R} \mathrm{Map}_{\mathcal{D}}(RLc, Rd) \xrightarrow{\eta} \mathrm{Map}_{\mathcal{D}}(c, Rd)$$

are inverse to each other. The second part also goes through as usual but requires some Yoneda technology. □

Proposition 4.10. *If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence then it is left and right adjoint to its inverse.*

Proof. DO THIs! □

Facts:

- Given $L : \mathcal{C} \to \mathcal{D}$ then thepair (R, η) is unique if it exists.
- The composite of left adjoints is again left adjoint to the composite of the the right adjoints. Coherently one can say that for the ∞-category of ∞-categories and left adjoint repspectively right adjoint functors we have:
$$\mathrm{Cat}_\infty^L \xrightarrow{\sim} (\mathrm{Cat}_\infty^R)^{\mathrm{op}}$$
- Let $\mathcal{I}$ be a small ∞ category consider the constant functor map:
$$\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$$
Assume that Δ has a left adjoint L. Then for every functor $FL\mathcal{I} \to \mathcal{C}$ we get form the unit of the adjunction a map:
$$F \to \Delta(LF)$$
such that the induced map:
$$\mathrm{Map}_{\mathcal{C}}(LF, y) \xrightarrow{\sim} \mathrm{Map}(F, \delta(y)$$
is an equivalence. Hence by definition this means that:
$$LF = \operatorname*{colim}_{\mathcal{I}} F$$

Proposition 4.11. *If $\mathcal{C}$ has all $\mathcal{I}$-shaped colimits then Δ admits a left adjoint.*

- *Assume that $L : \mathcal{C} \to \mathcal{D}$ has for any d and object Rd and a map:*
$$LRd \to d$$
such that:
$$\mathrm{Map}_{\mathcal{C}}(c, Rd) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(Lc, d)$$
is an equivalence, then L admits a right adjoint R given pointwise by $R(d)$.

Exercise 4.12.

- $\mathcal{S} \xrightarrow{\pi_0} \mathrm{Set}$ *is a left adjoint*
- $\mathcal{C}/X \to C$ *is a left adjoint provided that $\mathcal{C}$ has products*
(Check!)

Proposition 4.13. *Left adjoint functors preserve colimits and right adjoint functors preserve limits.*

Proof. This should follow from the fact that composition of left adjoints are left adjoints (Check!) □

Definition 4.14. Given a functor $P : \mathcal{C} \to \mathcal{C}^{'}$ and a functor $F : \mathcal{C} \to \mathcal{D}$, then a triangle:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle P}\big\downarrow & \overset{\eta}{\Nearrow} & \nearrow_{\mathrm{Ran}F} \\ \mathcal{C}^{'} & & \end{array}$$

Is said to exhibit $\mathrm{Ran}F$ as the right Kan extension F along P if it is terminal in the sense given for derived functors.

$$\mathrm{Map}(H, \mathrm{Ran}F) \xrightarrow{\sim} \mathrm{Map}(H \circ P, F)$$

Corollary 4.15. *If the right Kan extension* $\mathrm{Ran}F$ *exists for everty* $F : \mathcal{C} \to \mathcal{D}$ *then the restriction functor :*

$$P^* : \mathrm{Fun}(\mathcal{C}^{'}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, D)$$

admits a right adjoint also denoted Ran

5 Non-abelian derived functors

Previously $F : \mathcal{A} \to \mathcal{B}$ additive, then we had two steps:

1. $K(F) : K\mathcal{A} \to K(\mathcal{B})$
2. $LF = \mathrm{Ran}(K(F))$

Assume that $K(\mathcal{A})$ has enough K-projectives, then we have:

1. $K(\mathcal{A})^{proj} \overset{i}{\hookrightarrow} K(\mathcal{A}) \overset{p}{\to} D(\mathcal{A})$ is an equivalence and i is left adjoint to the projection p
2. for any $F : K(\mathcal{A}) \to \mathcal{D}$ the left derived functor $LF : D(\mathcal{A}) \to \mathcal{D}$ under this identification is given by the restriction of F along i.
3. In particular $LF : D(\mathcal{A}) \to D(\mathcal{B})$ preserves connective objects i.e.:

$$LF(D(\mathcal{A}))_{\geq 0}) \subseteq D(\mathcal{B})_{\geq 0}$$

<u>Questions</u>:

– How can we universally characterize $p \circ K(F) : K(\mathcal{A})_{\geq 0} \to D(\mathcal{B})$?

– What if F is not additive?

Exercise 5.1. *Given a pointed functor* $F : \mathcal{A} \to \mathcal{B}$ *between abelian categories, if there exists a functor* $K(\mathcal{A})_{\geq 0} \to D(\mathcal{B})$ *given on chain complexes representing objects by applying* F *levelwise, then* F *must already be additive.*

5.1 Yoneda Lemma

We fix three Grothendieck universes:

$$\{\text{small sets}\} \subset \{\text{large sets}\} \subset \{\text{very large sets}\}$$

Then we have a notion of categories and ∞-categories in all of these: A small ∞-categories is a small simplicial set i.e. small set of objects, maps etc are small. Similarly for large and very large. Now $\mathcal{S}$ is the large ∞-category of small spaces and denote $\widehat{\mathcal{S}}$ the very large ∞-category of large spaces ($\mathcal{S} \subset \widehat{\mathcal{S}}$)

- Now for a large ∞-category $\mathrm{Map}_{\mathcal{C}}(a,b) \in \widehat{\mathcal{S}}$
- $\mathrm{Cat}_{\infty}^{small}$ the large ∞-category of small ∞-categories
- Cat_{∞} the very large ∞-category of large ∞-categories
- $\mathcal{C} \in \mathrm{Cat}_{\infty} \implies \mathcal{C}[W^{-1}] \in \mathrm{Cat}_{\infty}$
- $\mathrm{Map}_{D\mathcal{A}}(X,Y) \simeq \underrightarrow{\mathrm{colim}}_{\widehat{X}\to X} \mathrm{Map}(\widehat{X},Y) \in \widehat{\mathcal{S}}$

Definition 5.2. We say that a large ∞-category $\mathcal{C}$ is *locally small* if for any pari $a,b \in \mathcal{C}$ the space:

$$\mathrm{Map}_{\mathcal{C}}(a,b) \in \widehat{\mathcal{S}}$$

is equivalent to an object in $\mathcal{S}$ i.e. it is essentially small. In this case the functor:

$$\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \widehat{\mathcal{S}}$$

factors through $\mathcal{S} \subset \widehat{\mathcal{S}}$

Construction: For any large ∞-category $\mathcal{C}$ we have a functor:

$$j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}}) = \widehat{P}(\mathcal{C})$$

$$c \mapsto \underline{c} = \mathrm{Map}_{\mathcal{C}}(-,c)$$

called the *Yoneda embedding.* Moreover if $\mathcal{C}$ is locally small then this factors as:

$$j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) = P(\mathcal{S})$$

Theorem 5.3. *Yoneda Lemma*

1. *The functor j is fully faithful*
2. *For any $F : \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ and any $x \in \mathcal{C}$ there is a natural equivalence:*

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\underline{x}, F) \simeq F(x)$$

3. *Every object $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}})$ is a (large) colimit of objects of the form $\underline{x}$ for $x \in \mathcal{C}$*

Proof. ad (3): For a given $F : \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ consider the pullback of ∞-categpries:

$$\begin{array}{ccc} \mathcal{C}_F & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}})/F \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{j} & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}}) \end{array}$$

Then $\mathcal{C}/F$ is large. Now we claim that:

$$F \simeq \underset{x \in \mathcal{C}/F}{\mathrm{colim}}\, \underline{x}$$

□

Now if $\mathcal{C}$ is small, then every functor $F : \mathcal{C} \to \mathcal{S}$ is a small colimit of representables
Now let $F : \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{C}$ is small and $\mathcal{D}$ possibly large.

Proposition 5.4. *In this setup assume that $\mathcal{D}$ admits all small colimits, then:*

1. *There is an essentially unique colimit preserving functor such that we have a commutative diagram of ∞-categories:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle j}\big\downarrow & \nearrow & \\ P(\mathcal{D}) & & \end{array}$$

   (This also defines a left Kan extension)

2. *If $\mathcal{D}$ is locally small then this is left adjoint to the restricted Yoneda embedding:*

$$D \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{F^*} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) = P(\mathcal{C})$$

Corollary 5.5. *We have that* $\mathrm{Fun}(P(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *i.e.* $P(\mathcal{C})$ *is the universal ∞-category obtained from $\mathcal{C}$ by freely adjoining small colimits.*

<u>Construction</u>: Let K be any class of small colimit shapes e.g. all colimits, finite colimits, filtered, geometric realizations and so on. We form $P^K(\mathcal{C}) \subset P(\mathcal{C})$ as the smallest full subcategory which contains representables and which is closed under K-indexed colimits.

Proposition 5.6. *We have for any large ∞-category $\mathcal{D}$ which admits K-index colimits that restriction along $j : \mathcal{C} \to P^K(\mathcal{C})$ is an equivalence:*

$$\mathrm{Fun}^K(P^K(\mathcal{C}), \mathcal{D}) \xrightarrow{j^*} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where the left term is the category of K-indexed colimit preserving functors and the inverse is given by left Kan extension.

Example 5.7. For $\mathcal{C}$ any ∞-category we have:

$$\mathrm{Ind}(\mathcal{C}) = P^{\text{filtered}}(\mathcal{C})$$

In particular if $\mathcal{C}$ is a 1-category then so is $\mathrm{Ind}(\mathcal{C})$

Proposition 5.8. *Assume that $\mathcal{A}$ has enough compact projective objects. Then we have an equivalence:*

$$D(\mathcal{A})_{\geq 0} \simeq K(\mathcal{A}^{\mathrm{proj}})_{\geq 0} \simeq P^{\Delta^{\mathrm{op}}\,\mathrm{filt}}(\mathcal{A}^{\mathrm{cp}}) \simeq \mathrm{Fun}^{\Pi}((\mathcal{A}^{\mathrm{cp}})^{\mathrm{op}}, \mathcal{S})$$

where the rightmost term are finite product preserving functors.

Corollary 5.9. *A functor:*

$$D(\mathcal{A})_{\geq 0} \to \mathcal{E}$$

that preserves geometric realizations and filtered colimits is uniquely determined by its restriction to the subcategory of compact projectives $\mathcal{A}^{\mathrm{cp}}$

Definition 5.10. Let $\mathcal{C}$ be an ∞-category, then $x \in \mathcal{C}$ is called *compact* if $\mathrm{Map}_{\mathcal{C}}(x, -)$ commutes with geometric realizatons. Moreover if $\mathcal{C}$ is a 1-category it is called compact if $\mathrm{Hom}_{\mathcal{C}}(x, -)$ commutes with split coequalizers.

Definition 5.11. Let $\mathcal{C}$ be an ordinary category which admits small colimits and is generated under small colimits by $\mathcal{C}^{\mathrm{cp}}$. Then the animation $\mathrm{Ani}(\mathcal{C})$ is defined as:

$$P^{\Delta,\mathrm{filt}}(\mathcal{C}^{\mathrm{op}}) \subseteq \mathrm{Fun}^{\prod}((\mathcal{C}^{\mathrm{cp}}, \mathcal{S})$$

Example 5.12. $\mathrm{Ani}(\mathrm{Set}) \simeq \mathcal{S}$

6 HKR and the Cotangent Complex

Recall: Start with a 1-category $\mathcal{C}$ which is generated by compact projective objects (where $K \in \mathcal{C}$ was called projective if $\mathrm{Hom}_{\mathcal{C}}(K, -)$ preserves filtered colimits and reflexive coequalizers. That $\mathcal{C}$ is generated by these means that $\mathrm{Hom}_{\mathcal{C}}(K, -)$ detects isomorphisms) Then we get an ∞-category:

$$\mathrm{Ani}(\mathcal{C}) = \mathrm{Fun}^{\Pi}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathcal{S})$$

This is the ∞-category freely generated from $\mathcal{C}^{\mathrm{op}}$ under filtered colimits and geometric realizations. More precisely:

- $\mathcal{C}^{\mathrm{op}} \subseteq \mathrm{Ani}(\mathcal{C})$ full subcategory
- $\mathrm{Ind}(\mathcal{C}^{\mathrm{cp}}) \subseteq \mathrm{Ani}(\mathcal{C})$
- An arbitrary object $X \in \mathcal{C}$ can be represented as a geometric realization of a simplicial diagram in $\mathrm{Ind}(\mathcal{C}^{\mathrm{cp}})$
- For $X \in \mathcal{C}^{\mathrm{cp}}$ we have that:

$$\mathrm{Map}_{\mathrm{Ani}(\mathcal{C})}(X, \operatorname*{colim}_{j \in \Delta^{\mathrm{op}}} Y_j) = \operatorname*{colim}_{j \in \Delta^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Ani}(\mathcal{C})}(X, Y_j)$$

$D(\mathcal{A})_{\geq 0} \simeq \mathrm{Ani}(\mathcal{A})$
For a not necessarily additive functor $F : \mathcal{A} \to \mathcal{B}$ we can compose with the Yoneda embedding $\mathcal{B} \to \mathrm{Ani}(\mathcal{B})$. This uniquely extends to a functor whihc preserves fileterd colimits and geometric realizations:

$$\mathrm{Ani}(\mathcal{A}) \to \mathrm{Ani}(\mathcal{B})$$

Example 6.1. Consider the functor:

$$\Lambda^n_R : \mathrm{Mod}_R \to M_R$$

gives a derived functor:

$$L\Lambda^n_R : D(R)_{\geq 0} \to D(R)_{\geq 0}$$

which is computed by representing objects by simplicial diagrams of projective modules and then applying the functor levelwise

Exercise 6.2. *Show that:* $L\Lambda^2_{\mathbb{Z}}(\mathbb{Z}/n, y) \simeq \mathbb{Z}/n[1]$

We have that:

$$\mathrm{ani(cRing)} = \mathrm{Fun}^{\Pi}((\mathrm{cRin}^{\mathrm{cp}})^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}((\mathrm{Poly}^{\mathrm{fg}})^{\mathrm{op}}, \mathcal{S})$$

$$\mathrm{Ind}(\mathrm{cRing}^{\mathrm{cp}}) = \mathrm{Poly}$$

So an object is represented by a simplicial diagram of polynomial rings and we have:

$$\mathrm{Map}_{\mathrm{Ani(cRing)}}(\mathbb{Z}[x], \operatorname*{colim}_{\Delta^{\mathrm{op}}} Y_i) = \mathrm{colim}_{\Delta^{\mathrm{op}}} Y_i$$

Example 6.3. The functor

$$HH(-/\mathbb{Z}) : \mathrm{Poly} \to D(\mathbb{Z})_{\geq 0}$$

commutes with filtered colimits and so it extends to a functor :

$$LHH : \mathrm{Ani(cRing)} \to D(\mathbb{Z})_{\geq 0}$$

which on $R \in \mathrm{cRing}$ agrees with $HH(R/\mathbb{Z})$ (In fact it agrees for all animated rings by considering the associated DGA).

Construction: For $C \in D(\mathbb{Z})$ we have a map $\tau_{\geq n}C \to C$ which is an iso on H_i for $\geq n$ and the object C has $H_i(\tau_{\geq n}C) = 0$ for all $i < n$. In fact this defines a functor:

$$\tau_{\geq n} : D(\mathbb{Z}) \to D(\mathbb{Z})$$

Exercise 6.4. *Construct the functor* $\tau_{\geq n}$ *using an adjunction.*

We also have maps $\tau_{\geq n+1}C \to \tau_{\geq n}C$ and one sees that the cofiber of this map is equivalent to the Eilenberg MacLane space $H_n(C)[n]$.
For $R \in \text{Poly}$ by what we've already shown we see that:

$$\begin{array}{ccc} \tau_{\geq n+1}HH(R/Z) & & \\ \downarrow & & \\ \tau_{\geq n}HH(R/\mathbb{Z}) & \xrightarrow{\text{cofib}} & \Omega^n_{R/\mathbb{Z}}[n] \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ \tau_{\geq 0}HH(R/\mathbb{Z}) & & \end{array}$$

Definition 6.5. We define

$$F^n_{HKR}HH(-,\mathbb{Z}) : \text{Ani}(\text{cRing}) \to D(\mathbb{Z})_{\geq 0}$$

as the nonabelian derived functor of $\tau_{\geq n}HH(-/\mathbb{Z})$

Then we get a cofiber sequence:

$$F^{n+1}_{\text{HKR}}(HH(R/\mathbb{Z})) \to F^n_{\text{HKR}}HH(R/\mathbb{Q}) \to L\Omega^n_{R/\mathbb{Z}}[n]$$

Lemma 6.6. *We have that* $F^n_{\text{HKR}}HH(R/\mathbb{Z}) \in D(\mathbb{Z})_{\geq n}$

Proof. This follows immediately since $D(\mathbb{Z})_{\geq n}$ is closed under colimits. □

Lemma 6.7. *If* $F : \text{cRing} \to^{\text{ab}}$ *commutes with reflexive coequalizers then:*

$$H_0(LF(R)) = F(R)$$

Proof. Resolve R by a simplicial diagram of polynomial rings $R_\bullet \to R$, then in fact:

$$R = \text{coeq}(R_0 \substack{\rightarrow\\ \leftarrow \\ \rightarrow} R_0)$$

Since $LF(R)$ is the complex associated to $F(R_\bullet)$ the homology $H_0(LF(R))$ is precisely this coequalizer. □

Exercise 6.8. *Show that the following functors commute with reflexive coequalizers:*

1. $\text{cRing} \to \text{Set},\ R \mapsto R^n$
2. $\text{cRing} \to \text{Ab},\ R \mapsto \mathbb{Z}[R^n]$
3. $\Omega^n_{-/\mathbb{Z}} : \text{cRing} \to \text{Ab}$

Theorem 6.9. *HKR Version 2 If* $L\Omega^n_{R/\mathbb{Z}}$ *has homology concentrated in degree 0 for each* n*, then HKR holds for* R*.*

Proof. The long exact sequence on homology associated to the cofiber sequence:

$$F^{n+1}_{\mathrm{HKR}} \to F^n_{\mathrm{HKR}} \to Ł\Omega^n_{R/\mathbb{Z}}[n]$$

shows that:

$$H_n(F^n_{\mathrm{HKR}}) \xrightarrow{\sim} \Omega^n_{R/\mathbb{Z}}$$

is an isomorphism and furthermore in higher degrees:

$$H_i(F^{n+1}_{\mathrm{HKR}}) \xrightarrow{\sim} H_i(F^n_{\mathrm{HKR}})$$

is an isomorphism for $i > n$. Hence we get equivalences:

$$\Omega^n_{R/Z} \xrightarrow{\sim} H_n(F^n_{\mathrm{HKR}}) \xrightarrow{\sim} H_n F^0_{\mathrm{HKR}} = HH_n(R/\mathbb{Z})$$

□

Proposition 6.10. *We have that $L\Omega^n_{R/\mathbb{Z}}$ agrees with the value on R of:*

$$\mathrm{Ani}(\mathrm{cRing}_{/R}) \to D(R)_{\geq 0} \to D(\mathbb{Z})_{\geq 0}$$
$$A \in \mathrm{Poly}_{/R} \mapsto R \otimes_A \Omega^n_{A/\mathbb{Z}}$$

Using this we see that:

$$R \otimes_A \Omega^n_{A/\mathbb{Z}} \cong \Lambda^n_R(R \otimes_A \Omega^1_{A/\mathbb{Z}})$$

Exercise 6.11. *Show that $A \mapsto R \otimes_A \Omega^1_{A/k}$ takes compact projctive objects in $k\mathrm{Alg}_{/R}$ to compact projective objects in Mod_R*

Thus we have that:

$$L\Omega^n_{R/\mathbb{Z}} = L\Lambda^n_R(L\Omega^1_{R/\mathbb{Z}})$$

Proposition 6.12. *If $L\Omega^1_{R/\mathbb{Z}}$ has homology concentrated in degree 0 and $\Omega^1_{R/\mathbb{Z}}$ is a flat R-moudle that $L\Omega^n_{R/\mathbb{Z}}$ is also concentrated in degree 0*

Proof. If $\Omega^1_{R/\mathbb{Z}}$ is finitely generated + projective then:

$$L\Lambda^n_R(\Omega^1_{R/\mathbb{Z}}) \simeq \Lambda^n_R \Omega^1_{R/\mathbb{Z}}[0]$$

In general use Lazard's theorem, i.e. every flat R-module is a filtered colimit of finitely generated projective ones. □

Theorem 6.13 (HKR Final Version)**.** *If $L\Omega^1_{R/\mathbb{Z}}$ has homology concentrated in degree 0 and $\Omega^1_{R/\mathbb{Z}}$ is a flat R-module then:*

$$HH_n(R/\mathbb{Z}) \simeq \Omega^n_{R/\mathbb{Z}}$$

Remark 6.14. One can replace $\mathbb{Z}$ with some commutative base ring k and everything works the same.

7 The Cotangent Complex and Obstruction Theory

We defined $L\Omega^1_{-/k}$ by taking the non-abelian derived functor:

$$\Omega^1_{-/K} : k\mathrm{Alg} \to \mathrm{Ab}$$

and hence it defines a functor:

$$\mathrm{Ani}(k\mathrm{Alg}) \to D(\mathbb{Z})_{\geq 0}$$

Lemma 7.1. *The following agree:*

1. $L\Omega_{-/k} : \mathrm{Ani}(k\mathrm{Alg}) \to D(\mathbb{Z})_{\geq 0}$ *evaluated on* R

2. $L(A \mapsto R \otimes_A \Omega^1_{A/k}) : \mathrm{Ani}(k\mathrm{Alg}_{/R} \to D(R))_{\geq 0}$ *evaluated on* $R \xrightarrow{\mathrm{id}} R$

Proof. Write R as $\mathrm{colim}_{i\in\Delta^{\mathrm{op}}} A_i$ for A_i levelwise a polynomial ring. Then the second expression is given by:

$$\begin{aligned}
\operatorname*{colim}_{i\in\Delta^{\mathrm{op}}} R \otimes_{A_i} \Omega^1_{A_i/k} &\simeq \operatorname*{colim}_{i\in\Delta^{\mathrm{op}}} \left(\operatorname*{colim}_{j\in\Delta^{\mathrm{op}}} A_j \right) \otimes^L_{A_i} \Delta^1_{A_i/k} \\
&\simeq \operatorname*{colim}_{i\in\Delta^{\mathrm{op}}} \left(\operatorname*{colim}_{j\in\Delta^{\mathrm{op}}_{i/}} A_j \right) \otimes^L_{A_i} \Omega^1_{A_i/k} \\
&\simeq \operatorname*{colim}_{(i\to j)\in(\Delta^{\mathrm{op}})^{\Delta^1}} A_j \otimes_{A_i} \Omega^1_{A_i/k} \\
&\simeq \operatorname*{colim}_{i\in\Delta^{\mathrm{op}}} \Omega^1_{A_i/k} = L\Omega^1_{R/k}
\end{aligned}$$

□

The functor:

$$\mathrm{Ani}(k\mathrm{Alg}_{/}R) \to D(R)_{\geq 0}$$

Nowe also preserves finite coproducts and ,since by definition it already preserves filtered colimits and geometric realizations, hence it preserves all colimits.

Example 7.2. For $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ for a regular sequence f_i then:

$$\begin{array}{ccc}
k[f_1, \dots, f_m] & \longrightarrow & k[x_1, \dots, x_n] \\
\downarrow & & \downarrow \\
k & \longrightarrow & R
\end{array}$$

is a pushout in $\mathrm{Ani}(k\mathrm{Alg}_{/R})$ hence the functor $L(R \otimes_{-} \Omega^1_{-/k}$ takes this diagram to the pushout in $D(R)_{\geq 0}$

$$\begin{array}{ccc}
R\{df_1, \dots df_n\} & \xrightarrow[(\partial_i f_j)_{i,j}]{} & R\{dx_1, \dots, dx_n\} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L\Omega^1_{R/k}
\end{array}$$

Where the horizontal map is the Jacobi Matrix. Hence we have that $L\Omega^1_{R/K}$ has H_0 given by the cokernel of the Jacobi and H_1 given by the kernel.

Definition 7.3. For a commutative k-algebra R we call $L\Omega^1_{R/k}$ the *Cotangent Complex* of R and denote it $L_{R/k}$

Given a commutative k-algebra S and an S-module M we have the *split square-zero extension* denoted $S \oplus M$. Then one can consider the lifting problem:

$$\begin{array}{ccc}
 & & S \oplus M \\
 & \nearrow & \downarrow \\
R & \xrightarrow{\phi} & S
\end{array}$$

Such a lift corresponds to a ϕ-linear derivation $R \to M$ i.e. an R-module map:

$$\Omega^1_{R/k} \to \phi_* M$$

where $\phi_* M$ denotes the restriction of scalars.

Proposition 7.4. *1. There is a functor:*

$$\begin{aligned} D(S)_{\geq 0} &\to \mathrm{Ani}(k\mathrm{Alg})_{/S} \\ P &\mapsto S \oplus P \qquad \text{for } P \text{ projective} \end{aligned}$$

2. We have an equivalence of mapping spaces:

$$\mathrm{Map}_{D(R)_{\geq 0}}(L_{R/k}, \phi_* M) \simeq \mathrm{fib}_\phi \left(\mathrm{Map}_{\mathrm{Ani}(k\mathrm{Alg})}(R, S \oplus M) \to \mathrm{Map}_{\mathrm{Ani}(k\mathrm{Alg})}(R, S) \right)$$

A surjective map of commutative k-algebras $\widetilde{R} \to R$ with kernel I satisfying $I^2 = 0$ is called a (not necessarily split) *square zero extension.* ow $L_{R/\widetilde{R}}$ has $H_0 = 0$ and $H_1 = R \otimes_{\widetilde{R}} I = I$. We have a tautological map of R-modules:

$$L_{R/\widetilde{R}} \to I[1]$$

inducing an isomorphism on H_1 corresponding to a map of animated $\widetilde{R}$-algebras:

$$R \xrightarrow{\delta} R \oplus I[1]$$

Proposition 7.5. *The diagram:*

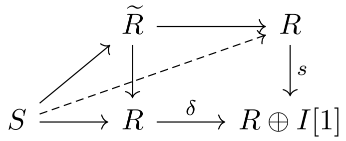
$$\begin{array}{ccc} \widetilde{R} & \longrightarrow & R \\ \big\downarrow & \lrcorner & \big\downarrow s \\ R & \xrightarrow[\delta]{} & R \oplus I[1] \end{array}$$

is a pullback of animated $\widetilde{R}$-algebras.

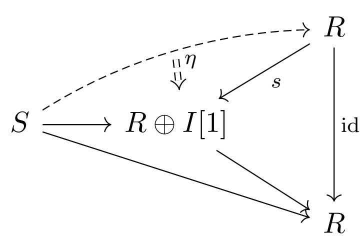
In summary: Square zero extensions $I \to \widetilde{R} \to R$ of k-algebras correspond to maps:

$$L_{R/k} \to I[1]$$

Moreover maps $S \to \widetilde{R}$ lifting a given map $S \to R$ correspond to lifts:



or equivalently a lift in the diagram:



Where we see that the map $R \to R$ is already determined, and hence the datum of such a diagram is precisely the homotopy η. Thus since $R \xrightarrow{s} R \oplus I[1]$ corresponds to the map $0 \to I[1]$ in, such a lift is the same as a nullhomotopy of the induced map:

$$L_{S/k} \to I[1]$$

The set of homotopy classes of nullhomotopies form a *torsor* over $\mathrm{Map}(L_{S/k}, I[1])$ Altogether:

– Square zero extenstions $I \to \widetilde{R} \to R$ are classified by:

$$\pi_o \mathrm{Map}_{D(A)}(L_{R/k}, I[1])$$

– For $S \to R$ there is an obstruction in

$$\pi_0 \mathrm{Map}_{D(S)}(L_{S/k}, I[1])$$

for the existence of lifts $S \to \widetilde{R}$

– Lifts (if they exist) are parametrized by:

$$\pi_1 \mathrm{Map}_{D(S)}(L_{S/k}, I[1]) \simeq \pi_0 \mathrm{Map}(L_{S/k}, I)$$

This is the classical statement that the lifts are a torsor under the group of derivations.

Exercise 7.6. *For a k-algebra S the following are equivalent:*

1. *For every square zero extension $\widetilde{R} \to R$ and every map $S \to R$ there is a lift $S \to \widetilde{R}$*
2. *$H_1 L_{S/k} = 0$ and $H_0 L_{S/k}$ is a projective S-module.*

Theorem 7.7. *If $R_1/\mathbb{F}_p$ is a perfect F_p-algebra then there exist flat $\mathbb{Z}/p^n$-algebras R_n, unique up to isomorphism, with $R_1 \simeq R_n \otimes_{\mathbb{Z}/p^n} \mathbb{F}_p$.*

In particular $R_{n-1} \simeq R_n \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^{n-1}$ and hence we get a diagram

$$\cdots \to R_n \to \cdots \to R_2 \to R_1$$

with limit $R := \varprojlim_n R_n$ the unique flat, p-complete $\mathbb{Z}_p$-algebra with:

$$R_1 \simeq R \otimes_{\mathbb{Z}_p} \mathbb{F}_p = R/p$$

Proof. We can characterize R_n as a non-split square zero extension of R_{n-1} by R_1. Namely by tensoring the exact sequence:

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to 0$$

over $\mathbb{Z}/p^n$ with R_n obtaining:

$$0 \to R_1 \to R_n \to R_{n-1} \to 0$$

These are classified by

$$\pi_0 \mathrm{Map}_{D(R_{n-1})}(L_{R_{n-1}/\mathbb{Z}}, R_1[1])$$

which need to be compatible with an element of

$$\pi_0 \mathrm{Map}_{D(\mathbb{Z}/p^{n-1})}(L_{\mathbb{Z}/p^{n-1}/\mathbb{Z}}.\mathbb{Z}/p[1])$$

classifying $\mathbb{Z}/p^n \to /Zp^{n-1}$

Lemma 7.8. *Isomorphism classes of such R_n are a torsor over the group:*

$$\pi_0 \mathrm{Map}_{D(R_{n-1})}(L_{R_{n-1}/\mathbb{Z}/p^{n-1}}, R_1[1]) \simeq \pi_0 \mathrm{Map}_{D(R_1)}(L_{R_1/\mathbb{F}_p}, R_1[1])$$

Lemma 7.9. *If $R_1/\mathbb{F}_p$ is perfect then $L_{R_1/\mathbb{F}_p} \simeq 0$*

Proof. The map:

$$\begin{aligned} \phi : A &\to A \\ x &\mapsto x^p \end{aligned}$$

induces the 0-map on $\Omega^1_{A/\mathbb{F}_p}$ (Exercise). By resolving R_1 via polynomial rings and using the naturality of the Frobenius we see that ϕ acts by 0 on $L_{R_1/\mathbb{F}_p}$, but since A was perfect it also acts by an isomorphism. □

Hence there is a unique such lift and we are done. □

The R form the theorem is called the (p-typical) *Witt vectors* of R_0 written $W(R_0)$.

Example 7.10. For $\mathbb{F}_{p^n}/\mathbb{F}_p$ lifts to a flat $\mathbb{Z}_p$-algebra $W(\mathbb{F}_{p^n})$ with $W(\mathbb{F}_{p^n})/p \simeq \mathbb{F}_p$.

8 Hochschild Homology of Schemes

Throughout this section X is a scheme over a ring k. We want to define $HH(X/k)$. There are two ways to do this:

1. Extend from the affine case
2. Generalize $HH(-/k)$ to dg-categories over k and define:

$$HH(X/k) = HH(\mathrm{Perf}(X)/k)$$

   The "non -commutative" approach.

We will adapt the first approach.

Definition 8.1. We define:

$$HH(X/k) = \lim_{U \subseteq X \text{ open affine}} HH(\mathcal{O}(U)/k) \in D(k)$$

I.e. the limit over the functor:

$$HH(\mathcal{O}(-)/k) : \{U \subseteq X \text{ affine open }\}^{\mathrm{op}} \to D(k)$$

Note that:

$$HH(\mathrm{Spec}(R)/k) = \lim_{U \subseteq \mathrm{Spec}(R)} HH(\mathcal{O}(U)/k) = HH(\mathcal{O}(\mathrm{Spec}(R))/k) = HH(R/k)$$

since $\mathrm{Spec}(R)$ is initial in the category of affine opens.

Example 8.2. Consider the projective space $\mathbb{P}^1_k$, this has an open cover:

$$(\mathbb{A}^1_k)^+ \cup (\mathbb{A}^1_k)^- = \mathbb{P}^1_k$$

with intersection:

$$(\mathbb{A}^1_k)^+ \cap (\mathbb{A}^1_k)^- = \mathbb{G}_m$$

Do we have some descent/Mayer-Vietoris principle?

Theorem 8.3. *For any pair $U, V \subset X$ open such that $X = U \cap V$ the square:*

$$\begin{array}{ccc} HH(X/k) & \longrightarrow & HH(U/k) \\ \downarrow & & \downarrow \\ HH(V/k) & \longrightarrow & HH(U \cap V/k) \end{array}$$

is a pullback in the derived category $D(k)$ i.e. Hochschild Homology satisfies Zariski descent.

Remark 8.4. We will see that $HH(-/k)$ satisfies even fpqc descent.

Corollary 8.5. *In the situation of the theorem we get a long exact sequence:*

$$\begin{array}{ccccc} & & \dots & \longrightarrow & HH_{n+1}(U \cap V) \\ HH_n(X) & \longrightarrow & HH_n(U) \oplus HH_n(V) & \longrightarrow & HH_n(U \cap V) \\ HH_{n-1}(X) & \longrightarrow & \dots & & \end{array}$$

Exercise 8.6. *For any commutative ring R and $x \in R$ we have that:*

$$HH(R[[x^{-1}]/k]) \simeq HH(R/k) \otimes_R R[x^{-1}] = HH(R/k)[x^{-1}]$$

Example 8.7. We have for $\mathbb{P}^!_k$ the pullback square:

$$\begin{array}{ccccc}
HH(\mathbb{P}^1/k) & \longrightarrow & HH(k[x]/k) & \qquad & x \\
\downarrow & & \downarrow & & \downarrow \\
HH(k[y]/k) & \longrightarrow & HH(k[x^{\pm}]/k) & & x
\end{array}$$

$$y \longmapsto x^{-1}$$

And hence the Mayer Vietoris sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & HH_1(\mathbb{P}^1/k) & \longrightarrow & k[x]dx \oplus k[y]dy & \longrightarrow & k[x^{\pm}]dx \\
& & HH_0(\mathbb{P}^1/k) & \longrightarrow & k[x] \oplus k[y] & \longrightarrow & k[x^{\pm}] \\
& & HH_{-1}(\mathbb{P}^1/k) & \longrightarrow & 0 & &
\end{array}$$

With the map:

$$\begin{aligned} k[x] \oplus k[y] &\to k[x^{\pm}] \\ (f,g) &\mapsto f(x) + f(x^{-1}) \end{aligned}$$

which is clearly surjective, hence $HH_{-1}(\mathbb{P}^1/k) = 0$. Moreover we have the map:

$$\begin{aligned} k[x]dx \oplus k[y]dy &\to k[x^{\pm}]dx \\ (fdx, gdy) &\mapsto f(x)dx + \underbrace{g(x^{-1})d(x^{-1})}_{=-\frac{g(x^{-1})dx}{x^2}} \end{aligned}$$

Which has a one-dimensional cokernel and trivial kernel. Hence we have $HH_0(\mathbb{P}^1/k) = k \oplus k$ and $HH_1(\mathbb{P}^1/k) = 0$

Proof of the Theorem. Have $U, V \subseteq X$ open and want to show that we have a pullback:

$$\begin{array}{ccc}
\lim_{A \subseteq X} HH(A) \simeq HH(X) & \longrightarrow & HH(U) \simeq \lim_{A \subseteq X} HH(A \cap U) \\
\downarrow & & \downarrow \\
\lim_{A \subseteq X} HH(A \cap V) \simeq HH(V) & \longrightarrow & HH(U \cap V) \simeq \lim_{A \subseteq X} HH(A \cap U \cap V)
\end{array}$$

Hence since limits commute with limits we can assume that X is affine. Moreover one can assume that U, V are affine and standard open (not so clear). Thus we reduce to the case:

$$\begin{array}{ccc}
HH(R/k) & \longrightarrow & HH(R[x^{-1}]/k) \\
\downarrow & & \downarrow \\
HH(R[y^{-1}]) & \longrightarrow & HH(R[x^{-1}, y^{-1}]/k)
\end{array}$$

For this we get the diagram:

$$\begin{array}{ccc} HH(R/k) & \longrightarrow & HH(R/k)\otimes_R R[x^{-1}] \\ \downarrow & & \downarrow \\ HH(R/k)\otimes_R R[y^-1] & \longrightarrow & HH(R/k)\otimes_R R[x^{-1},y^{-1}] \end{array}$$

To see that this is a pullback it suffices to prove that:

$$\begin{array}{ccc} R & \longrightarrow & R[x^{-1}] \\ \downarrow & & \downarrow \\ R[y^{-1}] & \longrightarrow & R[x^{-1},y^{-1}] \end{array}$$

is a pullback. However its clearly a pushout and hence a pullback by stability. □

Recall: If R has flat cotangent complex (i.e. cotangent complex concentrated in degree 0 given by a flat R-module) then we have that:

$$HH_*(R/k)\simeq \Omega^*_{R/k}$$

In general we had a filtration $F^*_{\mathrm{HKR}}HH(R/k)$ with n-th associated graded is given by:

$$F^{n+1}_{\mathrm{HKR}}\to F^n_{\mathrm{HKR}}\to L\Omega^n_{R/k}[n]=(L\Lambda^n)(L_{R/k})[n]$$

This filtration is complete in the sense that:

$$\varprojlim F^*_{HKR}=0\in D(k)$$

We define a similar filtration on $HH(X/k)$ via:

$$F^n_{\mathrm{HKR}}(X/k):=\lim_{U\subseteq X\text{ affine open}} F^n_{\mathrm{HKR}}(HH(\mathcal{O}(U)/k))$$

Proposition 8.8. *This defines a complete filtration on $HH(X/k)$ i.e.we have that:*

$$\varprojlim_n F^n_{\mathrm{HKR}}(X/k)=0$$

And the n-th associated graded is given by:

$$\lim_{U\subseteq X} L\Omega^n_{U/k}[n]=R\Gamma(X;L\Omega^n_{\mathcal{O}/k}[n])$$

Which is called the derived Hodge cohomology of X.

Remark 8.9. Note that if X is smooth we have $L\Omega=\Omega$ i.e.:

$$R\Gamma(X;\Omega^n_{\mathcal{O}/k}[n])$$

is just the sheaf cohomology of the sheaf of Kähler differentials, which is known as Hodge cohomology.

Proof. The completeness is clear since limits commute with limits. The second claim follows since cofibers commute with limits. □

In particular we get a spectral sequence:

$$R^i\Gamma(X;L\Omega^n_{\mathcal{O}/k}[n])\implies HH_*(X/k)$$

Corollary 8.10. *If $\mathbb{Q}\subseteq k$ then:*

$$HH(X/k)\simeq\prod R\Gamma(X;L\Omega^n_{\mathcal{O}/k}[n])$$

Proof. We have seen this in the case $X=\mathrm{Spec}(k[x_1,\dots,x_n])$. Hence it follows for any affine X by non-abelian derivation. The general case follows(?). □

9 Spectra

We write $\mathcal{S}_*$ for the category of pointed spaces. We have two functors:

$$\Sigma, \Omega : \mathcal{S}_* \to \mathcal{S}_*$$

defined via:

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

Definition 9.1. A *Spectrum* is a sequence of spaces X_i where $i \geq 0$ together with equivalences $X_i \xrightarrow{\sim} \Omega X_{i+1}$. We define the category of spectra as:

$$\mathrm{Sp} := \mathrm{Eq}(\mathcal{S})_* \times_{\mathcal{S}_*} \mathrm{Eq}(\mathcal{S}_*) \times_{\mathcal{S}_*} \dots$$

where $\mathrm{Eq}(\mathcal{S}_*)$ is the full subcategory of $\mathcal{S}_*^{\Delta^1}$ consisting of those maps which are equivalences. The pullback is formed along the maps:

$$\begin{array}{ccc} & & \mathrm{Eq}(\mathcal{S}_*) \\ & & \downarrow{\scriptstyle \Omega\circ\mathrm{source}} \\ \mathrm{Eq}(\mathcal{S}_*) & \xrightarrow[\mathrm{target}]{} & \mathcal{S}_* \end{array}$$

Note that objects Sp are indeed spectra. Moreover a map consists of a list of maps $X_i \xrightarrow{f_i} Y_i$ and choices of homotopies in the squares:

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow & \swArrow & \downarrow \\ \Omega X_{i+1} & \xrightarrow[\Omega f_{i+1}]{} & \Omega Y_{i+1} \end{array}$$

Remark 9.2. Alternatively we have that:

$$\mathrm{Sp} \simeq \lim(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*) \in \mathrm{Cat}_\infty$$

and moreover:

$$\mathrm{Map}_{\mathrm{Sp}}(X, Y) \simeq \lim(\mathrm{Map}_{\mathcal{S}_*}(X_i, Y_i))$$

Example 9.3.
- $(HA)_i = K(A, i)$ with equivalences $K(A, i-1) \xrightarrow{\sim} \Omega K(A, i)$
- $X \in \mathcal{S}_*$ the *Suspension spectrum* of X is given by:

$$(\Sigma^\infty X)_i = \operatorname*{colim}_k \Omega^k \Sigma^{k+i} X$$

  in particular $\Sigma^\infty S^0 =: \mathbb{S}$ is the *Sphere Spectrum*

Lemma 9.4. *We have an equivalence:*

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, Y_0)$$

Proof sketch. We have $X \to Y_0 \simeq \Omega^{k+i} Y_{k+i}$ is adjoint to $\Sigma^{k+i} X \to Y_{k+i}$. Now applying Ω^k we get a map:

$$\Omega^k \Sigma^{k+i} X \to Y_i$$

inducing a map on the colimit:

$$(\Sigma^\infty X)_i = \operatorname*{colim}_k \Omega^k \Sigma^{k+i} X \to Y_i$$

□

Definition 9.5. We define the *infinite loop space functor* as:

$$\begin{aligned} \Omega^\infty :& \mathrm{Sp} \to \mathcal{S}_* \\ & Y \mapsto \Omega^\infty Y = Y_0 \end{aligned}$$

Thus the previous lemma says that we have an adjunction $\Sigma^\infty \dashv \Omega^\infty$

Example 9.6. For $K \in \mathcal{S}_*$ *finite* we have that:

$$\begin{aligned} \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty K, \Sigma^\infty X) &\simeq \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^\infty \Sigma^\infty X) \\ &= \mathrm{Map}_{\mathcal{S}_*}(K, \operatorname*{colim}_k \Omega^k \Sigma^k X) \\ &\simeq \operatorname*{colim}_k \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^k \Sigma^k X) \\ &\simeq \operatorname*{colim}_k \mathrm{Map}_{\mathcal{S}_*}(\Sigma^k K, \Sigma^k X) \end{aligned}$$

In particular for $K = S^0$ we get that:

$$\pi_n \mathrm{Map}(\mathbb{S}, \Sigma^\infty X) = \pi_n^s X$$

Lemma 9.7. Sp *has all limits, filtered colimits and a zero object.*

Proof. A zero object is given by $X_i = *$. Limits and filtered colimits commute with Ω since it is a pullback. Thus by the limit formula we gave for mapping spaces these exist and are computed pointwise. □

Lemma 9.8. *The functor* $\Omega : \mathrm{Sp} \to \mathrm{Sp}$ *is an equivalence.*

Proof. Since limits are computed pointwise takes a spectrum $(X_0, X_1, \dots)$ to $(\Omega X_0, X_0, X_1, \dots)$, thus the inverse is given by shifting in the other direction. □

Proposition 9.9. Sp *has all colimits and pushout squares are pullback squares and vice versa.*

Proof. Any ∞-category with finite limits, a zero object and Ω is an equivalence has pushouts and these are determined by the fact that pushout squares = pullback squares (Omitted). Once we have pushouts we are done since we already have filtered colimits. □

Note that $\Sigma : \mathrm{Sp} \to \mathrm{Sp}$ is inverse to Ω it is given by shifting to the left. In partiuclar it is *not* computed pointwise. Moreover note that we have:

$$Y_i = \Omega^\infty \Sigma^i Y$$

Example 9.10. We have that:

$$[\Sigma^\infty X, \Sigma^n HA]_{\mathrm{Sp}} = [X, K(A,n)]_{\mathcal{S}_*} = H^n(X; A)$$

and similarly:

$$[\Sigma^n \mathbb{S}, Y]_{\mathrm{Sp}} = [S^n, \Omega^\infty]_{\mathcal{S}_*} = \pi_n(\Omega^\infty Y)$$

Lemma 9.11. *For every $Y \in \mathrm{Sp}$ we have that:*

$$Y \simeq \operatorname*{colim}_i \Sigma^{-i}\Sigma^\infty$$

where we denote $\Sigma^{-i} = \Omega^i$

Proof. We have that:

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}}(Y, Y) &\simeq \lim_i \mathrm{Map}_{\mathcal{S}_*}(Y_i, Z_i)\\ &\simeq \lim_i \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty Y_i, \Sigma^i Z)\\ &\simeq \lim_i \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{-i}\Sigma^\infty Y_i, Z)\\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(\operatorname*{colim}_i \Sigma^{-i}\Sigma^\infty, Z)\end{aligned}$$

□

<u>Construction</u>: Denote the singular chains of a space X by $C_*(X)$ then we have a functor:

$$\begin{aligned}\mathrm{Sp} &\to D(\mathbb{Z})\\ Y &\mapsto \operatorname*{colim}_i C_*(Y_i)[-i]\end{aligned}$$

which preserves colimits and maps $\Sigma^\infty X$ to $C_*(X)$ so in particular $\mathbb{S} \mapsto \mathbb{Z}[0]$.

Definition 9.12. Let $\mathrm{Sp}^{\mathrm{fin}} \subseteq \mathrm{Sp}$ be the full subcategory of objects of the form:

$$\Sigma^{-n}\Sigma^\infty K \quad \text{with} \quad X \in \mathcal{S}\text{finite}$$

Exercise T. his is closed under finite colimits.

In fact this can be described as the subcategory generated by $\mathbb{S}$ under finite colimits. Moreover we have the following:

Proposition 9.13. Sp *is freely generated under filtered colimits by* $\mathrm{Sp}^{\mathrm{fin}}$ *i.e.* $\mathrm{Sp} = \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$

Proof. We get a functor from the universal property of Ind. Moreover $\mathrm{Sp}^{\mathrm{fin}}$ in Sp is compact since:

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}}(\Omega^\infty K, \underrightarrow{\operatorname{colim}}_j Y^j) &\simeq \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^\infty\Sigma^n \underrightarrow{\operatorname{colim}}_j Y^j)\\ &\simeq \underrightarrow{\operatorname{colim}}_j \mathrm{Map}_{\mathcal{S}_*}(K, \omega^\infty\Sigma^n Y^j)\end{aligned}$$

where we have used that filtered colimits are computed pointwise. Now write:

$$Y = \operatorname*{colim}_i \Sigma^{-i}\Sigma^\infty Y_i$$

and then each Y_i as a filtered colimit of finite spaces which is always possible. □

<u>Analogy</u>: Have $D^{\mathrm{perf}}(R) \subseteq D(R)$ the full subcategory of perfect complexes and we have:

$$D(R) = \mathrm{Ind}(D^{\mathrm{perf}}(R))$$

so we want to think of Sp as "$D(\mathbb{S})$" however additional subtleties arise from the fact that $\mathrm{Map}(\mathbb{S}, \mathbb{S})$ is not discrete. Moreover think of the singular chains functor $C_* : \mathrm{Sp} \to D(\mathbb{Z})$ as the "basechange along $\mathbb{S} \to \mathbb{Z}$"

Definition 9.14. $X \in \mathrm{Sp}$ is called n-connective if $\pi_i X = 0$ for $i < n$ and n-coconnective if $\pi_i X = 0$ for $i > n$ and denote the corresponding full subcategories as $\mathrm{Sp}_{\geq n}$ and Sp_{leqn} respectively.

Proposition 9.15. *The inclusion* $\mathrm{Sp}_{\leq n} \hookrightarrow \mathrm{Sp}$ *has a left adjoint denoted* $\tau_{\leq n}$.

Proof. $X_n = X$, buo;d X_i as follows:
Pick generators of $\pi_i(X_{i-1})$ which induce a map:

$$\bigoplus \Sigma^i \mathbb{S} \to X_{i-1}$$

and define X_i as the cofiber. Then set $\tau_{\leq n} X := \mathrm{colim}_i X_i$, then we have:

- $\tau_{\leq n} X \in \mathrm{Sp}_{\leq n}$ and $\pi_i \tau_{\leq n} X = \pi_i X$ by the long exact sequence on homotopy groups and using fact that $\mathbb{S}$ is connective.
- Every map $X \to Y \in \mathrm{Sp}_{\leq n}$ factor over these cofibers and thus over $\tau_{\leq n} X \to Y$

□

Definition 9.16. We denote by $\tau_{\leq n}$ be the fiber of the map $X \to \tau_{\leq n-1} X$

Lemma 9.17. *1. For* $X \in \mathrm{Sp}_{\geq n}$ *and* $Y \in \mathrm{Sp}_{n-1}$ *we have* $\mathrm{Map}(X, Y) \simeq 0$

2. We have an equivalence of categories $\mathrm{Sp}_{\geq n} \cap \mathrm{Sp}_{\leq n} \simeq \mathrm{Ab}$

Proof. Any map $X \to Y$ factors through $\tau_{\leq n-1} X \simeq 0 \to Y$ which shows 1. For 2. obeserve that π_n gives a functor and that mapping spaces are discrete since:

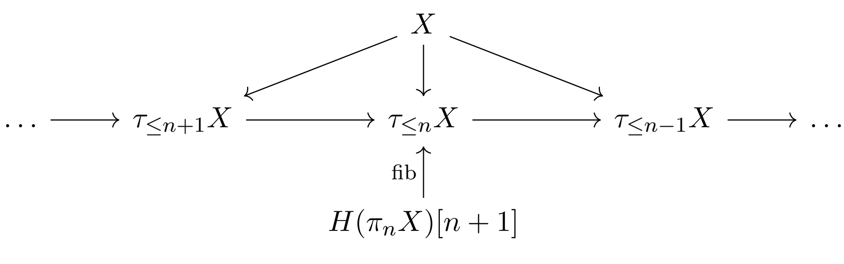
$$\pi_i \mathrm{Map}(X, Y) \simeq \mathrm{Map}(\Sigma^{\mathrm{i}}\mathrm{X}, \mathrm{Y}) \simeq 0 \quad \text{for } i > 0$$

If $\pi_n X$ is free then we can choose a map:

$$\bigoplus \Sigma^n \mathbb{S} \to X$$

which is an iso on π_n and thus an equivalences under $\tau_{\leq n}$. If it is not free chose a map $bigoplus \Sigma^n \mathbb{S} \to X$ which is surjective of π_n, then the fiber has free π_n. Use the exact sequence for $\pi_n \mathrm{Map}(-, Y)$ (??) □

<u>Postnikov Tower:</u>
For every X we have a tower:



10 Symmetric monoidal ∞-categories

<u>Recall</u>:

- An (ordinary) symmetric monoidal category consists of the following data:
  - $\mathcal{C}$ a category
  - $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
  - $1_{\mathcal{C}} \in \mathcal{C}$

– natural isomorphisms:

$$(a \otimes b) \otimes c \xrightarrow{\sim} a \otimes (b \otimes c)$$
$$a \otimes 1_{\mathcal{C}} \xrightarrow{\sim} a \xrightarrow{\sim} 1_{\mathcal{C}} \otimes a$$
$$a \otimes b \xrightarrow{\sim} b \otimes a$$

satisfying certain coherence conditions (e.g. Pentagon identity)

– We abbreviate this datum as $(\mathcal{C}, \otimes)$ and write:

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n := (((a_1 \otimes a_2) \otimes a_3) \otimes \dots) \otimes a_n$$

– A *lax symmetric monoidal functor* between symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is given by a functor $F : \mathcal{C} \to \mathcal{D}$ together with natural morphisms:

  – $F(c) \otimes_{\mathcal{D}} F(c') \to F(c \otimes_{\mathcal{C}} c')$ for $c, c' \in \mathcal{C}$
  – $1_{\mathcal{D}} \to F(1_{\mathcal{D}})$

  satisfying certain compatibility conditions

– A lax symmetric monoidal functor is called a (strong) symmetric monoidal functor if the maps are isomorphisms.

– A symmetric monoidal category $(\mathcal{C}, \otimes)$ is called *closed* if for every object $c \in \mathcal{C}$ the functor:

$$- \otimes c : \mathcal{C} \to \mathcal{C}$$

  admits a right adjoint denoted $\underline{\mathrm{Hom}}(c, -)$. In particular it preserves colimits in $\mathcal{C}$.

Proposition 10.1. *Let R be a commutative ring, then the category Mod_R admits an essentially unique closed symmetric monoidal structure with tensor unit R denoted:*

$$\otimes_R : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$$

Exercise 10.2. *Show this (Only the uniqueness?)*

Moreover for any map $R \to S$ of commutative rings the functor:

$$- \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$$

canonically refines to a symmetric monoidal functor since:

$$(M \otimes_R N) \otimes_R S \cong (M \otimes_R S) \otimes_S (N \otimes_R S)$$

<u>Goal today:</u>

1. Define symmetric monoidal ∞-categories such that for an ordinary symmetric monoidal category $\mathcal{C}$ the nerve $N(\mathcal{C})$ is an example.

2. Define lax/strong symmetric monoidal functors.

Theorem 10.3. *1. The ∞-categories Sp and $D(R)$ admit essentially unique closed symmetric monoidal structures with units $\mathbb{S}$ respectively R*

$$\otimes_{\mathbb{S}} : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp} \qquad \otimes_R^L : D(R) \times D(R) \to D(R)$$

2. The functors:

$$C_* : \mathrm{Sp} \to D(\mathbb{Z}) \qquad \Sigma^\infty : \mathcal{S} \to \mathrm{Sp} \qquad - \otimes_R^L S : D(R) \to D(R)$$

for any map $R \to S$ inherit canonical strong symmetric monoidal structures

Proof Sketch. For $X \in \mathrm{Sp}$ have:

- $\mathbb{S} \otimes X \cong X$
- $(\Sigma^\infty_+ Y) \simeq X \simeq \Sigma^\infty_+(\mathrm{colim}_Y *) \otimes X \simeq \mathrm{colim}_Y(\Sigma^\infty_+ * \otimes X) \simeq \mathrm{colim}_Y X$
- $(\Sigma^{-n} Y) \otimes X \simeq \Sigma^{-n}(Y \otimes X)$
- For a general spectrum Y we have $Y \simeq \mathrm{colim}(\Sigma^{\infty - n} Y_n)$
  Thus $Y \otimes X$ is determined up to equivalence.

□

Let Fin_* be the category of finite pointed sets. Every object is isomorphic to a set $\langle n \rangle = \{0, \dots, n\}$ with basepoint 0. For every $1 \leq i \leq n$ there is a map:

$$\rho^i : \langle n \rangle \to \langle 1 \rangle \quad k \mapsto \begin{cases} 1 & k = i \\ 0 & \text{else} \end{cases}$$

Definition 10.4. (1st version, Segal)
A symmetric monoidal category is a functor:

$$\underline{\mathcal{C} : N(\mathrm{Fin}_*) \to \mathrm{Cat}_\infty}$$

such that for every $\langle n \rangle$ the induced maps:

$$\underline{\mathcal{C}}(\langle n \rangle) \xrightarrow{(\rho^i_*)} \prod_{i=1}^n \mathcal{C}(\langle 1 \rangle)$$

are equivalences.

<u>Notation:</u> We write $\mathcal{C} = \mathcal{C}(\langle 1 \rangle)$ and:

$$\otimes : \mathcal{C} \times \mathcal{C} \xrightarrow{\sim} \mathcal{C}(\langle 2 \rangle) \xrightarrow{m_*} \mathcal{C}(\langle 1 \rangle) = \mathcal{C}$$

With m the map:

$$\begin{aligned} m : \langle 2 \rangle &\to \langle 1 \rangle \\ 0 &\mapsto 0 \\ 1 &\mapsto 1 \\ 2 &\mapsto 1 \end{aligned}$$

and unit:

$$1 : * = \mathcal{C}(\langle 0 \rangle) \to \mathcal{C}(\langle 1 \rangle) = \mathcal{C}$$

Exercise 10.5. *For a symmetric monoidal 1-category* $\mathcal{C}$ *construct a functor* $\mathrm{Fin}_* \to \mathrm{Cat}_\infty$ *which takes a pointed set* $S \coprod *$ *to* $\mathcal{C}^{\times S}$

Definition 10.6. A symmetric monoidal functor is given by a natural transformation $\underline{C} \to \underline{D}$. Hence one can define:

$$\mathrm{SymMonCat}_\infty \subseteq \mathrm{Fun}(N(\mathrm{Fin}_*, \mathrm{Cat}_\infty))$$

as a full subcategory.

Definition 10.7. (2nd, Lurie):
A symmetric monoidal ∞ category is a functor:

$$\mathcal{C}^\otimes \to N(\mathrm{Fin}_*)$$

satisfying the following conditions:

1. It is a cocartesian fibration.
2. The induced maps:
$$\mathcal{C}^{\otimes}_{\langle n\rangle} \xrightarrow{\rho^i_!} (\mathcal{C}^{\otimes}_{\langle 1\rangle})^{\times n}$$
are equivalences. We again write:
$$\mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1\rangle} \qquad \otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}^{\otimes} - \langle 2\rangle \xrightarrow{m_!} \mathcal{C}^{\otimes}_{\langle 1\rangle} = \mathcal{C}$$

Theorem 10.8 (Lurie)**.** *The two definitions are equivalent, that is for any such symmetric monoidal category $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ we get an induced functor:*
$$\begin{aligned} N(\mathrm{Fin}_*) &\to \mathrm{Cat}_\infty \\ \langle n\rangle &\mapsto \mathcal{C}^{\otimes}_{\langle n\rangle} \end{aligned}$$
and vice versa.

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal 1-category. We define $\mathcal{C}^{\otimes}$ as follows:

- Objects: $c_1, \ldots, c_n$ objects in $\mathcal{C}$, $\langle n\rangle \in \mathrm{Fin}_*$
- Morphisms from $C_1, \ldots, c_n$ to $d_1, \ldots, d_m$ are given by:
  - $\langle n\rangle \xrightarrow{f} \langle m\rangle$ a map in Fin_*
  - for each $k \in \langle m\rangle \setminus 0$ a map in $\mathcal{C}$:
$$\bigotimes_{i\in f^{-1}(k)} c_i \to d_k$$
- There is a canonical projection $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$

A morphism $f : \langle n\rangle \to \langle m\rangle$ is called *inert* if the induced map:
$$f^{-1}(\langle m\rangle \setminus 0) \to \langle m\rangle \setminus 0$$
is a bijection.

Definition 10.9. A *lax symmetric monoidal functor* between symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ is a functor:
$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{F^{\otimes}} & \mathcal{D}^{\otimes} \\ & \searrow \quad \swarrow & \\ & N(\mathrm{Fin}_*) & \end{array}$$
such that $F^{\otimes}$ sends coCartesian lifts of inert morphisms to coCartesian lifts.

Exercise 10.10. *Check that for two symmetric monoidal 1-categories a lax monoidal functor $\mathcal{C} \to \mathcal{D}$ induces a functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$*

Definition 10.11. Define a 1-categotry $\mathrm{Ass}^{\otimes}_{act}$ with:

- Objects: finite sets
- Morphisms: maps $S \to T$ together with a total ordering on each $f^{-1}(t) \subseteq S$

This is symmetric monoidal with respect to disjoint union.

Write $\langle 1\rangle$ for the 1-element set in $\mathrm{Ass}^{\otimes}_{act}$ is an associative algebra object:

$$\langle 1\rangle \coprod \langle 1\rangle \to \langle 1\rangle$$

$$\varnothing \to \langle 1\rangle$$

Definition 10.12. An associative algebra in a symmetric monoidal ∞-category $\mathcal{C}$ is given by a symmetric monoidal functor:

$$N(\mathrm{Ass}^{\otimes}_{act}) \to \mathcal{C}$$

the underlying object is the value at $\langle 1\rangle$

Exercise 10.13. *Check that the category of symmetric monoidal functors* $\mathrm{Ass}^{\otimes}_{act} \to \mathrm{Ab}$ *is equivalent to the category of rings.*

Definition 10.14. For a finite, non-empty totally ordered set S we define:

$$\mathrm{Cut}(S) := \left\{(S_0, S_1) \mid S_0, S_1 \subseteq S,\ S_0 < S_1,\ S_0 \coprod S_1 = S\right\}$$

$$\mathrm{Cit}^{cyc}(S) := {}^{\mathrm{Cut}(S)}\!\big/_{(S,\varnothing)\sim(\varnothing,S)}$$

Exercise 10.15. *Check that* $\mathrm{Cut} = \Delta^1$ *and* $\mathrm{Cut}^{cyc} = S^1 = \Delta^1/\partial\Delta^1$ *as functors* $\Delta^{\mathrm{op}} \to \mathrm{Set}$

In fact Cut^{cyc} defines a functor $\Delta^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}_{act}$, for a map $f : S \to T$ of totally ordered sets and a cut (S_0, S_1), the preimage $(f^*)^{-1}(S_0, S_1) \subseteq \mathrm{Cut}^{cyc}(T)$ is:

- If (S_0, S_1) is nontrivial then $(f^*)^{-1}(S_0, S_1)$ is given by all cuts "between" $f(S_0, S_1)$. The ordering is given by the natural one induced from S.
- If (S_0, S_1) is trivial then $(f^*)^{-1}(S_0, S_1)$ is given by all cuts "outside" of $f(S)$. Here the ordering starts with the elements $> f(S)$ and then goes around and continues with those $< f(S)$

Definition 10.16. For an algebra $A : \mathrm{Ass}^{\otimes}_{act} \to \mathcal{C}$ we define the Hochschild-Object of A:

$$HH(A/\mathcal{C}) := \operatorname*{colim}_{\Delta^{\mathrm{op}}}(A \circ \mathrm{Cut}^{cyc})$$

Lemma 10.17. *For an ordinary ring (or more genreally a dga), we have an algebra A in $D(\mathbb{Z})$. Then:*

$$HH(A/D(\mathbb{Z})) \simeq HH(A/\mathbb{Z})$$

Proof. The functor $\mathrm{Ch}(\mathbb{Z}) to D(\mathbb{Z})$ preserves tensor products of K-flat complexes. So if we take our dga to be K-flat the functor preserves all the tensor products in the cyclic bar complex. Then use that in $D(\mathbb{Z})$, a colimit of a simplicial diagram is computed by a total complex. ☐

Exercise 10.18. *For an ordinary ring R check that the composite:*

$$\Delta^{\mathrm{op}} \xrightarrow{\mathrm{Cut}^{\mathrm{cyc}}} \mathrm{Ass}^{\otimes}_{\mathrm{act}} \xrightarrow{R} \mathrm{Ab}$$

agrees with the cyclic Bar complex of R.

Definition 10.19. For a *ring spectrum* A (i.e. an associative algebra in Sp) we define the *Topological Hochschild Homology* of A as:

$$THH(A) := HH(A/\mathrm{Sp})$$

For an ordinary ring R we have the Eilenberg-MacLane spectrum HR which is canonically a ring spectrum. We write:

$$THH(R) = THH(HR)$$

Example 10.20. $THH(\mathbb{S}) = \mathbb{S}$

Definition 10.21. Define a 1-category $\mathrm{LMod}^{\otimes}_{\mathrm{act}}$ with::

- Objects: Finite sets with elements labeled (or "colored") by $\{a, m\}$ (i.e. a map $S \to \{a, m\}$)
- Morphisms: Maps $S \to T$ together with a total ordering on each fiber and:
  - the fiber of an a-colored element is completely a-colored.
  - the fiber of an m-colored element contains exactly one m-colored element which is also the maximum.

We also define $\mathrm{RMod}^{\otimes}_{\mathrm{act}}$ the same way but replacing maximum with minimum at the end. A left module in $\mathcal{C}$ is a symmtetric monoidal functor $\mathrm{LMod}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$
Moreover define a 1-category $\mathrm{LRMod}^{\otimes}_{\mathrm{act}}$ via:

- Objects finite colored sets as before with colors $\{r, a, l\}$
- maps $f : S \to T$ with a total ordering on each fiber and:
  - fibers of a-colored elements are a-colored
  - fibers of r-colored elements have exactly one r-colored elements which is the minimum and the rest a-colored
  - Same with l-colored but the maximum

A symmetric monoidal functor $\mathrm{LRMod}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ is a pair of functors $\mathrm{LMod}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ and $\mathrm{RMod}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ which agree on $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$

Definition 10.22. $N \otimes_A M$ is the colimit of the composite:

$$\Delta^{\mathrm{op}} \xrightarrow{\mathrm{Cut}(-)} \mathrm{LRMod}_{\mathrm{act}^{\otimes}} \xrightarrow{N,A,M} \mathcal{C}$$

Where we color $(\varnothing, S)$ as r, $(S, \varnothing)$ as l and all other cuts as a.

Remark 10.23. We can also do this with bimodules where we again have colors $\{a.m\}$ with fibers of m-colored elements have exactly one m-colored element with no further condition to obtain a category $\mathrm{BMod}^{\otimes}_{\mathrm{act}}$. Then $\mathrm{Cut}^{\mathrm{cyc}}$ factors through this and we can make define a version of Hochschild Homology with coefficients in a bimodule $HH(A/\mathcal{C}; M)$.

Definition 10.24. $\mathrm{Comm}^{\otimes}_{\mathrm{act}} = \mathrm{Fin}$ so a commutative algebra object in an ∞-category $\mathcal{C}$ is a functor $\mathrm{Fin} = \mathrm{Comm}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$

Lemma 10.25. *For a commutative algebra A the object $HH(A/\mathcal{C})$ again has a commutative algebra structure.*

Proof. Omitted □

There is a functor $H : D(\mathbb{Z}) \to \mathrm{Sp}$ such that $\pi_*(H(C) = H_*(C))$ which is lax symmetric monoidal and preserves colimits. Thus we get a natural map:

$$THH(HR) \to H(HH(R))$$

in partiucal a map:

$$THH_*(HR) \to HH_*(R)$$

In fact H caconically factors through an equivalence of ∞-categories:

$$D(\mathbb{Z}) \xrightarrow{\sim} \mathrm{Mod}(H\mathbb{Z})$$

which is symmetric monoidal. Hence we have:

$$H(HH(R)) = HH(HR/\mathrm{Mod}(H\mathbb{Z})) = THH(HR/H\mathbb{Z})$$

Example 10.26. If R is a $\mathbb{Q}$-algbra then $THH_*(R) \to HH_*(R)$ is an isomorphism. This follows since:

$$H\mathbb{Q} \simeq \mathbb{S} \otimes H\mathbb{Q} \simeq H\mathbb{Z} \otimes H\mathbb{Q}$$

so in particular:

$$\mathbb{S} \otimes HR \simeq H\mathbb{Z} \otimes HR$$

and thus:

$$HR \otimes_{\mathbb{S}} HR \simeq HR \otimes_{H\mathbb{Z}} HR$$

by the explicit construction of the Bar complex.

Proposition 10.27. *For an ordinary ring R the map:*

$$THH_i(R) \to HH_i(R)$$

is an isomorphism for $i \leq 2$ and surjective for $i = 3$

Proof. The fiber of the map:

$$THH(HR) \to THH(HR/H\mathbb{Z}))$$

is the geometric realization of a simplicial diagram of the form:

$$\begin{array}{c} \vdots \\ \downarrow\downarrow\downarrow \\ \mathrm{fib}(HR \otimes_{\mathbb{S}} HR \to HR \otimes_{H\mathbb{Z}} HR) \\ \downarrow\downarrow \\ \mathrm{fib}(HR \to HR) \end{array}$$

The first term is clearly zero. The following terms are 2-connective (follows from the analysis of the connectivity of the map $\mathbb{S} \to H\mathbb{Z}$) and thus the realization is 3-connective. □

For $\mathbb{F}_p$ we see that:

$$THH_2(\mathbb{F}_p) \cong HH_2(\mathbb{F}_p) \cong \mathbb{F}_p$$

with a generator denoted x.

Theorem 10.28 (Böckstedt)**.** *We have that:*

$$THH_*(\mathbb{F}_p) = \mathbb{F}_p[x]$$

Note that the map:

$$\mathbb{F}_p[x] \to F_p\langle x\rangle$$

is zero in degrees $\geq 2p$

11 $\mathbb{E}_n$-algebras

$\mathcal{C}$ a symmetric monoidal ∞-category i.e.:

1. $\mathcal{C}$ a symmetric monoidal 1-category $\implies$ $N(\mathcal{C})$ symmetric monoidal ∞-cat
2. $\mathcal{C}$ a topologically/simplicially enriched symmetic monoidal category $\implies$ $N_\Delta(\mathcal{C})$ is a symmetric monoidal ∞-category
3. The category of spaces $\mathcal{S}$ with the tensor product given by the product

4. The category of pointed spaces $\mathcal{S}_*$ with the smash product.

Recall:

- An associative algebra in $\mathcal{C}$ is a symmetric monoidal functor:

$$N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \to \mathcal{C}$$

The category of such algebras is:

$$\mathrm{Alg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}, \mathcal{C})$$

- A commutative algebra in $\mathcal{C}$ is given by a symmetic monoidal functor:

$$N(\mathrm{Fin}_*) \to \mathcal{C}$$

with the category denoted as:

$$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(\mathrm{Fin}_*, \mathcal{C})$$

Warning: In a 1-category $\mathcal{C}$ we have that $\mathrm{CAlg}(\mathcal{C})$ is a full subcategory of $\mathrm{Alg}(\mathcal{C})$ i.e. being commutative is a property. This false in ∞-categories! There is a map induced from the functor $\mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathrm{Fin}_*$ which forgets the order on the preimage, but it's not fully faithful.

Definition 11.1. Let $0 \leq n < \infty$, we define a symmetric monoidal ∞-category $(\mathbb{E}_n^{\otimes})_{\mathrm{act}}$ as the homotopy coherent nerve of the topologically enriched with:

- Objects: $\coprod_k D^n$ with $D^n = (0,1)^n$
- Morphisms: $\coprod_k D^n \to \coprod_{k'} D^n$ are given by rectilinear embeddings, i.e. emebeddings of topological spaces which on each disk is rectilinear, i.e. given by an affine linear map with matrix of the form:

$$\mathrm{diag}(\alpha_1, \ldots, \alpha_n),\ \alpha_i \in \mathbb{R}_{>0}$$

This category is topologically enriched and and symmetric monoidal via disjoint union of disks.

Definition 11.2. An $\mathbb{E}_n$-algebra in $\mathcal{C}$ is a symmetric monoidal functor:

$$(\mathbb{E}^{\otimes}_{\mathrm{act}}) \to \mathcal{C}, \qquad \mathrm{Alg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}((\mathbb{E}_n)^{\otimes}_{\mathrm{act}}, \mathcal{C})$$

Mapping spaces in $(\mathbb{E}_n^{\otimes})_{\mathrm{act}}$ are determined by $\mathrm{Map}(\coprod_k D^n, D^n)$. There is a natural map:

$$\mathrm{Map}(\coprod_k D^n, D^n) \to \mathrm{Conf}_k(D^n) = \mathrm{Hom}_{\mathrm{inj}}(\coprod_k *, D^n)$$

given by evaluation at the center point.

Exercise 11.3. *This map is a homotopy equivalence.*

Thus in particular we get

$$rmMap(D^n, D^n) \simeq \mathrm{Conf}_1(D^n) = D^n \simeq *$$

$$\mathrm{Map}(D^n \coprod D^n, D^n) \simeq \mathrm{Conf}_2(D^1) \simeq S^{n-1}$$

Thus if we have a functor:

$$\underline{A} : (\mathbb{E}_n^{\otimes})_{\mathrm{act}} \to \mathcal{C}$$

for the underlying object $\underline{A}(D^n) = A$ we get a map:

$$S^{n-1} \to \mathrm{Map}(A \otimes A, A)$$

Proposition 11.4. *There is an equivalence of symmetric monoidal categories:*

$$N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{\simeq} (\mathbb{E}_1^{\otimes})_{\mathrm{act}}$$

sendiung $S \mapsto \coprod_S D^1$

Proof. 1. $(\mathbb{E}_1^{\otimes})_{\mathrm{act}}$ is essentially a 1-category, i.e. the mapping spaces are discrete. This is immediate since they are simply given by linear orderings of $D^1 = (0,1)$. Moreover it's equivalent to $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ for the same reason.

2. This (implicitly constructed) functor is symmetric monoidal.

3. One can show that a functor is an equivalence in the category of symmetric monoidal ∞-categories iff it is an equivalence of underlying categories by using that equivalences in functor categories are detected pointwise.

□

Corollary 11.5. $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \simeq \mathrm{Alg}(\mathcal{C})$

Exercise 11.6. *1. Work out what* $\mathbb{E}_0$*-algebras are.*

2. Work out what $\mathbb{E}_2$*-algebras in the* ∞*-category* Cat *of 1-categories are*

There are symmetric monoidal functors:

$$(\mathbb{E}_0^{\otimes})_{\mathrm{act}} \xrightarrow{\times D^1} (\mathbb{E}_1^{\otimes})_{\mathrm{act}} \xrightarrow{\times D^1} (\mathbb{E}_2^{\otimes})_{\mathrm{act}} \to \dots$$

So we get induced functors:

$$\dots \to \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$$

Definition 11.7. The ∞-category of $\mathbb{E}_\infty$-algebras is defined as the limit of this diagram in Cat_∞

Theorem 11.8. *We have an equivalence* $\mathrm{Alg}_{[E]_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})$

Proof.

$$\begin{aligned}\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \varprojlim_n \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) simeq \varprojlim_n \mathrm{Fun}^{\otimes}((\mathbb{E}_n)_{\mathrm{act}}) \\ \simeq \mathrm{Fun}^{\otimes}(\varinjlim_n (\mathbb{E}_n^{\otimes})_{\mathrm{act}}, \mathcal{C}) \\ \simeq \mathrm{Fun}^{\otimes}(N(\mathrm{Comm}^{\otimes}_{\mathrm{act}}), \mathcal{C})\end{aligned}$$

□

Exercise 11.9. *Complete the proof by showing that:*

$$\varinjlim_n (\mathbb{E}^{\otimes} - n)_{\mathrm{act}} \simeq N(\mathrm{Comm}^{\otimes} - \mathrm{act})$$

Let $(\mathcal{C}, \otimes)$ be a closed symmetric monoidal category, or weaker assume that $- \otimes c : \mathcal{C} \to \mathcal{C}$ commutes with filtered colimits and geometric realizations, and $\mathcal{C}$ has all colimits and limits.

Theorem 11.10. *1. The* ∞*-category* $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}),\ 0 \le n \le \infty$ *admits all limits and colimits and the functor:*

$$\mathrm{Alg}_{\mathbb{E}-n}(\mathcal{C}) \xrightarrow{\mathrm{ev}_{D^n}} \mathcal{C}$$

preserves filtered colimits, geometric realizations (i.e. sifted colimits) and all limits. Moreover it detects equivalences.

2. For $n = \infty$ *the coproduct is given by* $A \otimes B,\ A, B \in \mathrm{CAlg}(\mathcal{C})$

3. *The ∞-category $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ admits a symmetric monoidal structure such that the functor:*

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$$

admits a canonical refinement to a symmetric monoidal functor.

Exercise 11.11. *Work out what $\mathbb{E}_n$-algebras in the 1-category* Ab *are and check the statement of the theorem there.*

Example 11.12. Consider $(\mathcal{S}, \times)$, then $X \in \mathrm{Alg}(\mathcal{S})$ is a "monoid in $\mathcal{S}$". Then $\pi_0(X)$ is an actual monoid in Set. We call X grouplike if $\pi_0 X$ is a group $\iff$ the map $X \times X \to X \times X$ sending $(a, b) \mapsto (ab, b)$ is a homotopy equivalence. An $\mathbb{E}_n$ algebra is called grouplike if the underlying $\mathbb{E}_1$-algebra is.
For every pointed space $X \in \mathcal{S}_*$ the n-fold loop space:

$$\Omega^n X = \mathrm{Map}_*(S^n, X) = \mathrm{Map}_*((D^n, \partial D^n), (X, *))$$

admits a canonical $\mathbb{E}_n$-algebra structure in $(\mathcal{S}, \times)$

Exercise 11.13. *Construct and $\mathbb{E}_n$-algebra structure on $\Omega^n X$, i.e. a symmetric monoidal functor:*

$$(\mathbb{E}_n^{\otimes})_{\mathrm{act}} \to \mathcal{S}$$

by giving an explicit functor between the topologically enriched categories.

Theorem 11.14. (Boardman-Vogt)
The functor:

$$\Omega^n : \mathcal{S}_*^{n-\mathrm{conn}} \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$$

is fully faithful with essential image given by the grouplike $\mathbb{E}_n$-algebras for any $0 \leq n < \infty$.

Proof. For $n = 1$ the functor:

$$\mathcal{S}_*^{\mathrm{conn}} \xrightarrow{\Omega} \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{grp}}(\mathcal{S})$$

Has an inverse given by the Bar complex construction constructed in th previous lecture. In fact we have naturally:

$$\Omega \mathrm{Bar}(G) \simeq G \qquad \mathrm{Bar}(G) = BG$$

Theorem 11.15. (Dunn-Additivity
For a symmetric monoidal ∞-category $\mathcal{C}$ we have:

$$\mathrm{Alg}_{\mathbb{E}_{n+m}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})), \quad m, m \geq 0$$

□

Corollary 11.16. *For any $\mathbb{E}_n$-algebra $A \in \mathcal{S}$ and any connected, pointed space $X \in \mathcal{S}_*^{\mathrm{conn}}$ we have that:*

$$\mathrm{Map}_{\mathbb{E}_n}(\Omega^n(\Sigma^n X), A) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, A)$$

i.e. $\Omega^n \Sigma^n X$ is the free $\mathbb{E}_n$-algebra on X!

Proof. We can consider the subspace $A' \subseteq A$ consisting of the unit components $\pi_0(A') \subseteq \pi_0(A)$. Then:

$$\mathrm{Map}_{\mathbb{E}_n}(\Omega^n \Sigma^n X, A) \simeq \mathrm{Map}_{\mathbb{E}_n}(\Omega^n \Sigma^n X, A')$$

Similarly since X is connected:

$$\mathrm{Map}_{\mathcal{S}_*}(X, A) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, A')$$

Thus we may assume that A is grouplike as well i.e. $A \simeq \Omega^n Y$ and hence:

$$\mathrm{Map}_{\mathbb{E}_n}(\Omega^n \Sigma^n X, \Omega^n Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(\Sigma^n X, Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, \Omega^n Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, A)$$

□

Corollary 11.17. *Assume that $\mathcal{C}$ has geometric realizations and $\otimes$ commutes with them in both variables separately. The Bar-construction defines a functor:*

$$\mathrm{Bar} : \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})_{/1} \to \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C})$$

Proof. Again follows since $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ using that the statement holds for $n = 1$ □

12 Böckstedt periodicity

We want to prove the following:

Theorem 12.1. (Böckstedt)

$$THH_*(\mathbb{F}_p) \cong \mathbb{F}_p[x] \quad , \ |x| = 2$$

We discuss a more structured but ultimately equivalent version of this:

Theorem 12.2. Böckstedt, Version 2
$THH(\mathbb{F}_p)$ is as an $\mathbb{E}_1$-algebra over $G\,\mathbb{F}_p$, free on one generator of degree 2, i.e.:

$$HH(\mathbb{F}_p) \simeq H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega S^3$$

Indeed for any other $\mathbb{E}_1$-algebra R we have:

$$\begin{aligned}\mathrm{Map}_{\mathbb{E}_1/H\,\mathbb{F}_p}(H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega S^3, R) &\simeq \mathrm{Map}_{\mathbb{E}_1}(\Omega S^3, \Omega^\infty R)\\ &\simeq \mathrm{Map}_{\mathbb{S}_*}(S^2, \Omega^\infty R)\\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}^2, R) = \pi_2 R\end{aligned}$$

Let us first see why these are equivalent:

Proof. The element $x \in THH_2(\mathbb{F}_p)$ we get from $HH(\mathbb{F}_p)$ defines an $\mathbb{E}_1$ map:

$$H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega S^3 \to THH(\mathbb{F}_p)$$

which is an equivalence if and only if $\pi_* THH(\mathbb{F}_p) = \mathbb{F}_p[X]$ since:

$$\pi_*(H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega S^3) \simeq H_*(\Omega S^3; \mathbb{F}_p) \simeq \mathbb{F}_p[x]$$

where the Homology carries the Pontryagin product.

Exercise 12.3. *Show the first iso!*

□

Lemma 12.4. *We have that:*

$$THH(R) \simeq R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} R$$

Proof sketch. Write $R \simeq R \otimes_R R = |\mathrm{Bar}(R)|$. Then:

$$R \otimes_{R\otimes_{\mathbb{S}} R^{\mathrm{op}}} R = \operatorname*{colim}_{\Delta^{\mathrm{op}}} R \otimes_{R\otimes_{\mathbb{S}} R^{\mathrm{op}}} R^{\otimes \bullet+1} = \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{Bar}^{\mathrm{cyc}}(R) = THH(R)$$

□

Before we go further we need to understand $H\,\mathbb{F}_p \otimes_{\mathbb{S}} H\,\mathbb{F}_p$ the *Dual Steenrod Algebra*

Theorem 12.5. (Milnor) *As an algebra we have:*

$$\pi_*(H\,\mathbb{F}_p \otimes_{\mathbb{S}} G\,\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1, \zeta_2, \dots], \ |\zeta_i| = 2^i - 1 & \text{for } \ p = 2 \\ \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \dots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \dots], \ |\tau_i| = 2p^i - 1, \ |\xi_i| = 2p^i - 2 & \text{for } \ p \neq 2 \end{cases}$$

View $G\,\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$ as an $\mathbb{E}_\infty - H\,\mathbb{F}_p$-algebra via the "inclusion" on the right factor.

Lemma 12.6. *As an $\mathbb{E}_2$-$H\,\mathbb{F}_p$-algebra, $H\,\mathbb{F}_p \otimes_{\mathbb{S}} G\,\mathbb{F}_p$ is free on a generator of degree 1, i.e. we have an equivalence:*

$$H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega^2 S^3 \xrightarrow{\simeq} H\,\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$$

Proof of Böchstedt V2 using this.

$$\begin{aligned} THH(\mathbb{F}_p) &= H\,\mathbb{F}_p \otimes_{H\,\mathbb{F}_p \otimes_{\mathbb{S}} H\,\mathbb{F}_p} H\,\mathbb{F}_p \\ &\simeq H\,\mathbb{F}_p \otimes_{H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega^2 S^3} H\,\mathbb{F}_p \end{aligned}$$

moreover we have that:

$$H\,\mathbb{F}_p \simeq H\,\mathbb{F}_p \otimes \Sigma^\infty *$$

Hence since the functor $H\,\mathbb{F}_p \otimes \Sigma^\infty(-)$ is symmetric monoidal monoidal and preserves colimits we can also compute the original tensor expression by first taking the Bar-resolution in spaces:

$$\begin{aligned} &\simeq H\,\mathbb{F}_p \otimes \Sigma^\infty \mathrm{Bar}(*, \Omega^2 S^3, *) \\ &\simeq H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega S^3 \end{aligned}$$

□

Remark 12.7. Böckstedts theorem and our lemma are equivalent since a map $A \to B$ of connected $H\,\mathbb{F}_p$-algebras is an equivalence if and only if the map:

$$H\,\mathbb{F}_p \otimes_A H\,\mathbb{F}_p \to H\,\mathbb{F}_p \otimes_B H\,\mathbb{F}_p$$

is an equivalence. This we apply to the map:

$$H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega^2 S^3 \to H\,\mathbb{F}_p \otimes_{\mathbb{S}} H\,\mathbb{F}_p$$

which we get since the left hand side is free on one generator.

Exercise 12.8.

1. *Let A be a connected $H\,\mathbb{F}_p$-algebra. Show that a map $N \to M$ of connective A-modules is an equivalence if:*

$$H\,\mathbb{F}_p \otimes_A N \to H\,\mathbb{F}_p \otimes_A M$$

is an equivalence.

2. *Let $A \to B$ be a map of connected $H\,\mathbb{F}_p$-algebras with:*

$$H\,\mathbb{F}_p \otimes_A H\,\mathbb{F}_p \to H\,\mathbb{F}_p \otimes_B H\,\mathbb{F}_p$$

an equivalence. Show that then $H\,\mathbb{F}_p \otimes_A B \simeq H\,\mathbb{F}_p$ and that $A \to B$ is an equivalence.

Let p be odd, for an $\mathbb{E}_\infty$-$H\,\mathbb{F}_p$-algebra there are the *Dyer-Lashof operations*:

$$\begin{aligned} Q^i &: \pi_n(R) \to \pi_{n+2(p-1)i}(R) \\ \beta Q^i &: \pi_n(R) \to \pi_{n+2(p-1)i-1}(R) \end{aligned}$$

for each integer $i \in \mathbb{Z}$. Moreover for $|x| = n$ even we have $Q^{n/2}x = x^p$ and $Q^i x = 0$ for $i < n/2$ (also valid for n odd). Similarly $\beta Q^i x = 0$ for $i \leq n/2$.
$Q^{\frac{n}{2}}$ for n even and $Q^{\frac{n+1}{2}}x, \beta Q^{\frac{n+1}{2}}x$ for n odd are already defined for an $\mathbb{E}_2$algebra.

Theorem 12.9. (Dyer-Lashof, $p = 2$ Araki-Kudo)

$$H_*(\Omega^2 S^3; \mathbb{F}_p) = \Lambda(a, Q^1 a, Q^p Q^1 a, \dots) \otimes \mathbb{F}_p\left[(\beta Q^1)a, (\beta Q^p)Q^1 a, \dots\right] \quad |a| = 1$$

Theorem 12.10. (Steinberger)
On $\pi_*(H\,\mathbb{F}_p \otimes_{\mathbb{S}} \otimes H\,\mathbb{F}_p)$ we have:

$$\tau_i = Q^{p^{i-1}} Q^{p^{i-2}} \cdots Q^1 \tau_0$$
$$\xi_i = \beta Q^{p^{i-1}} Q^{p^{i-2}} \cdots Q^1 \tau_0$$

Proof of Lemma. We have that the map

$$\pi_1(H\,\mathbb{F}_p \otimes_{\mathbb{S}} H\,\mathbb{F}_p) \xrightarrow{\sim} \pi_1(H\,\mathbb{F}_p \otimes_{H\mathbb{Z}} H\,\mathbb{F}_p) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p$$

is an isomorphism.

Exercise 12.11. *Show this using that the map $\mathbb{S} \to H\mathbb{Z}$ is 1-connective*

hence we get a map of $\mathbb{E}_2$-$H\,\mathbb{F}_p$-algebras:

$$H\,\mathbb{F}_p \otimes \Sigma^\infty \Omega^2 S^3 \to H\,\mathbb{F}_p \otimes_{\mathbb{S}} H\,\mathbb{F}_p$$

which is an isomorphim on π_1, but both sides are generated in the same way by the $\mathbb{E}_2$-Dyer-Lashof operations. Hence it is an isomorphism on π_*. □

13 Properties of THH

For k a commutative ring spectrum and R a k-algebra i.e. $R \in \mathrm{CAlg}(\mathrm{Mod}_k)$then:

$$THH(R/k) = HH(R/\mathrm{Mod}_k)$$

in particular we have:

$$HH(R) \simeq THH(R/\mathbb{Z})$$

since $D(R) \simeq \mathrm{Mod}_{H\mathbb{Z}}$

Proposition 13.1. *The functor:*

$$THH : \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Sp}$$

is symmetric monoidal. Where a tensor product of algebras is just the ordinary tensor product. In particular we have that:

$$THH(A \otimes_{\mathbb{S}} B) \simeq THH(A) \otimes_{\mathbb{S}} THH(B)$$

More generally for a symmetric monoidal category which admits geometric realizations the functor:

$$HH(-/\mathcal{C}) : \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

is symmetric monoidal.

Proof. Consider the forgetful functor $U : \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Sp}$ as an object in the symmetric monoidal category $\mathrm{Fun}^{\otimes}(\mathrm{Alg}(\mathrm{Sp}, \mathrm{Sp})$. As such it is an algebra object (Represented by $\mathbb{S}$?) and in fact:

$$THH(-) \simeq HH(U/\mathrm{Fun}^{\otimes}(\mathrm{Alg}(\mathrm{Sp}), \mathrm{Sp}))$$

□

Corollary 13.2. *If R is an $\mathbb{E}_n$-ring spectrum then $THH(R)$ is $\mathbb{E}_{n-1}$.*

Proof. We have that:

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}) \simeq \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})) \xrightarrow{\mathrm{Alg}_{\mathbb{E}_{n-1}}(THH)} \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Sp})$$

□

In particular for $n = \infty$ we have that $THH(R)$ is also $\mathbb{E}_\infty$
Recall: For $A, B \in \mathrm{CAlg}(\mathrm{Sp})$ we know that $A \otimes_{\mathbb{S}} B$ is the coproduct in $\mathrm{CAlg}(\mathrm{Sp})$

Proposition 13.3. *For $R \in \mathrm{CAlg}(\mathrm{Sp})$ we have that:*

$$THH(R) = \operatorname*{colim}_{S^1}(R) = R^{\otimes S^1} \in \mathrm{CAlg}(\mathrm{Sp})$$

Where the colimit is taken over the constant diagram:

$$S^1 \to * \xrightarrow{R} \mathrm{CAlg}(\mathrm{Sp})$$

Exercise 13.4. *Prove this! (We already proved that $S^1 = \mathrm{Bar}^{\mathrm{cyc}}$ as a simplicial set I think)*

Warning: This colimit is different than the one taken in Sp. In fact the latter is given by:

$$R \otimes S^1 = R \otimes \Sigma^\infty S^1 \simeq R \otimes \Sigma R$$

Now we prove some base change formulas for THH:

Proposition 13.5. *1. For any lax symmetric monoidal funcotr $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $A \in \mathrm{Alg}(\mathcal{C})$ we get a natural map:*

$$HH(FA/\mathcal{D}) \to F(HH(A/\mathcal{C}))$$

2. If F is strong symmetric monoidal and preserves geometric realizations, then this map is an equivalence.

Example 13.6. 1. Consider $H : D(\mathbb{Z}) \to \mathrm{Sp}$ which is symmetric monoidal. Hence we get a map:

$$THH(R) \to H(HH(R/\mathbb{Z}))$$

2. If $k \to k'$ is a map of commutative ring spectra then the functor:

$$- \otimes_k k' \ \mathrm{Mod}_k \to \mathrm{Mod}_{k'}$$

is symmetric monoidal an preserves all colimits. Hence we have that:

$$THH(R/k) \otimes_k k' \simeq THH(R \otimes_k k'/k')$$

Recall that an ordinary $\mathbb{F}_p$-algebra K is called *perfect* if the Frobenius is an isomorphism. We have maps $Hk \to THH(k)$ and $THH(\mathbb{F}_p) \to THH(k)$ and hence on homotopy groups:

$$k \to THH_*(k)$$

$$\mathbb{F}_p[x] \to THH_*(k)$$

So we get a combined map:

$$Hk \otimes_{H\mathbb{F}_p} THH(\mathbb{F}_p) \to THH(k) \quad k[x] \to THH_*(k)$$

Theorem 13.7. (Böckstedt periodicity for perfect rings *For any perfect $\mathbb{F}_p$-algebra the map $k[x] \to THH_*(k)$ is an isomorphism. Equivalently $THH(k)$ is the free $\mathbb{E}_1$-algebra on x over Hk*

Proof. We use the fact that for k-perfect there is a commutative ring spectrum $\mathbb{S}_{W(k)}$ called the *spherical Witt vectors* such that:

$$\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} H\mathbb{F}_p \simeq k$$

Then we have that:

$$\begin{aligned} THH(k) = THH(\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} H\mathbb{F}_p) &\simeq THH(\mathbb{S}_{W(k)}) \otimes_{\mathbb{S}} THH(\mathbb{F}_p) \\ &\simeq \big(THH(\mathbb{S}_{W(k)}) \otimes_{\mathbb{S}} \mathbb{F}_p\big) \otimes_{\mathbb{F}_p} THH(\mathbb{F}_p) \\ &\simeq THH(k/\mathbb{F}_p) \otimes_{\mathbb{F}_p} THH(\mathbb{F}_p) \end{aligned}$$

Hence it suffices to prove that:

$$THH(k/\mathbb{F}_p) = HH(k/\mathbb{F}_p)$$

is equal to k i.e. $HH_*(k/\mathbb{F}_p) = 0$ for $* \geq 1$ □

Exercise 13.8. *Show this last claim.*

For k an $\mathbb{E}_\infty$-ring we have proven that $HH(k) \simeq k^{\otimes S^1}$ and hence we get a map $THH(k) \to k$ of $\mathbb{E}_\infty$-rings induced as:

$$THH(k) = \operatorname*{colim}_{S^1} k \to \operatorname*{colim}_{*} k = k$$

such that the composite $k \to THH(k) \to k$ is the identity.

Remark 13.9. Such a retract $THH(k) \to k$ does not exist in general if k is only $\mathbb{E}_n$ for $n < \infty$

Theorem 13.10. *If k is a commutative ring spectrum and R an associative k-algebra, then we have that:*

$$THH(R/k) \simeq THH(R) \otimes_{THH(k)} k$$

14 p-adic completion

Recall that an abelian group is called p-complete if it is complete (and separated) with respect to the p-adic topology (i.e. the nbhd basis given by $p^n A \subseteq A$). Equivalently the natural map:

$$A \xrightarrow{\sim} \varprojlim_n A/p^n =: A_p^\wedge$$

is an isomorphism.

Definition 14.1. – For $X \in \mathrm{Sp}$, $n \in \mathbb{Z}$ we define:

$$X/n := \mathrm{cofib}(X \xrightarrow{\cdot n} X)$$

Equivalently $X/n \cong X \otimes_{\mathbb{S}} \mathbb{S}/n$.

– If $n \mid m$ there is a natural map $X/m \to X/n$ obtained as the cofibre in:

$$\begin{array}{ccc} X & \xrightarrow{m/n} & X \\ {\scriptstyle \cdot m}\downarrow & & \downarrow{\scriptstyle \cdot n} \\ X & \xrightarrow{\mathrm{id}} & X \\ \downarrow & & \downarrow \\ X/m & \longrightarrow & X/n \end{array}$$

– For $X \in \mathrm{Sp}$ a spectrum we define:

$$X_p^\wedge := \varprojlim_n X/p^n$$

and get a canonical map $X \to X_p^\wedge$ induced from the projections.

– A spectrum X is called *p-complete* if the map $X \to X_p^\wedge$ is an equivalence. We denote by $\mathrm{Sp}_p^\wedge \subseteq \mathrm{Sp}$ be the full subcategory of p-complete Spectra.

Theorem 14.2. *1. For every spectrum X the spectrum $X_p^\wedge$ is p-complete.*

2. The functor $(-)_p^\wedge : \mathrm{Sp} \to \mathrm{Sp}_p^\wedge$ is left adjoint to the inclusion $\mathrm{Sp}_p^\wedge \hookrightarrow \mathrm{Sp}$

3. The ∞-category $\mathrm{Sp}_p^\wedge$ has all limits and colimits.

Proof. 1. – $\mathrm{Sp}_p^\wedge$ is closed under limits and finite colimits since the cofiber and filtered limit functors commute with these.

- Observe that for any X the spectrum X/p is p-complete. (Somewhat subtle in general the multiplication by p map is not nullhomotopic on X/p! For example happens on $\mathbb{S}/2$)
- Inductively using the fibre sequence:

$$X/p \xrightarrow{\cdot p^{n-1}} X/p^n \to X/p^{n-1}$$

see that all X/p^n are p-complete
- Putting these together we esee that $X_p^\wedge$ is p-complete.

2. From the first part we see that $(-)_p^\wedge$ is idempotent, hence we get a canonical map:

$$\mathrm{Map}(X_p^\wedge, Y) \to \mathrm{Map}(X, Y)$$

for which we can construct an inverse (This basically works for any idempotent with some caveats.)

3. Limits and finite colimits are clear. For an arbitrary diagram $I \xrightarrow{X} \mathrm{Sp}_p^\wedge$ is given by:

$$\operatorname*{colim}_I^{\mathrm{Sp}_p^\wedge} X_i = \left(\operatorname*{colim}_I^{\mathrm{Sp}} X_i \right)_p^\wedge$$

which one can check explicitly.

□

Exercise 14.3. *1. Show that a spectrum X is p-complete if and only if:*

$$\lim(\dots \xrightarrow{\cdot p} X \xrightarrow{\cdot p} X \xrightarrow{\cdot p} X) \simeq 0$$

2. Show that X/p is p-complete

Definition 14.4. A map $f : X \to Y$ of spectra is called a *p-dic equivalence* if the induced map:

$$f_p^\wedge : X_p^\wedge \to Y_p^\wedge$$

is an equivalence.

Note that if X, Y are p-complete then f is a p-adic equivalence iff it is an equivalence.

Proposition 14.5. *1. A map $f : X \to Y$ is a p-adic equivalence iff the map $f/p : X/p \to Y/p$ is an equivalence.*

2. For X, Y connective a map $f : X \to Y$ is a p-adic equivalence iff the map $X \otimes_{\mathbb{S}} \mathbb{F}_p \to Y \otimes_{\mathbb{S}} \mathbb{F}_p$ is an equivalence.

Exercise 14.6. *Show this (First part is easy, second part uses some Postnikov tower tricks)*

Proposition 14.7. *For A an abelian group, then if A is p-complete we get that $HA \in \mathrm{Sp}$ is p-complete. The converse fails in general, but if A has bounded order of p^∞-torsion then it also holds.*

Definition 14.8. And abelian group A is called derived p-complete if HA is p-complete.

Remark 14.9. There is also a notin of p-completeness in $D(Z)$. Then A is derived p-complete iff it is p-complete in $D(\mathbb{Z})$ i.e. $A \simeq \mathrm{R}\varprojlim \operatorname{cone}(A \xrightarrow{p^n} A) \in D(Z)$

Theorem 14.10. *1. A spectrum X is p-complete iff $\pi_n X$ is derived p-complete for ever n.*

2. Assume that $\pi_n X$ has bounded order of p^∞-torsion for each n, then $\pi_*(X_p^\wedge) = \pi_*(X)_p^\wedge$

Example 14.11. – we have that $\pi_*\mathbb{S}_p^\wedge = \begin{cases} \mathbb{Z}_p, & * = 0 \\ p^\infty \text{ torsion of } \pi_*(\mathbb{S}), & * > 0 \\ 0, & * < 0 \end{cases}$

– $(H\mathbb{Q})_p^\wedge = 0$ and $H\mathbb{Z}_p$ is p-complete

Monoidal properties:

– $\mathrm{Sp}_p^\wedge$ admits a closed symmetric monoidal structure given by:

$$X\widehat{\otimes}_p T := (X \otimes_{\mathbb{S}} Y)_p^\wedge$$

more precisely structur is uniquely characterized by the functor:

$$\mathrm{Sp} \xrightarrow{(-)_p^\wedge} \mathrm{Sp}_p^\wedge$$

being strong symmetric monoidal.

– IF R is an $\mathbb{E}_n$ -algebra, then so is $R_p^\wedge$ and the map $R \to R_p^\wedge$ is an $\mathbb{E}_n$-map exhibiting $R_p^\wedge$ as the initial $\mathbb{E}_n$-algebra under R. More precisely the p-completion is left adjoint to the inclusion $\mathrm{Alg}_{\mathbb{E}-n}(\mathrm{Sp}_p^\wedge) \subseteq \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$.,y

Question: Can we recover any spectrum X from its p-completions $X_p^\wedge$ at all primes p.

Recall: An abelian group is called rational if it is uniquely divisible $\iff$ it is a $\mathbb{Q}$-vector space $\iff$ the map $X \to X \otimes_{\mathbb{Z}} \mathbb{Q} = X_{\mathbb{Q}}$ is an iso.

Definition 14.12. A spectrum X is called rational if $\pi_n(X)$ is rational for each n. We denote by $\mathrm{Sp}_{\mathbb{Q}} \subseteq \mathrm{Sp}$ the full subcategory of rational spectra.

Theorem 14.13. *1. X is rational iff it admits the structure of an $H\mathbb{Q}$-module. (Which is actually unique up to contractible choice.)*

2. $X \otimes_{\mathbb{S}} \mathbb{Q} = X_{\mathbb{Q}}$ is rational and $\pi_*(X_{\mathbb{Q}}) = \pi_*(X)_{\mathbb{Q}}$

3. The functor $- \otimes_{\mathbb{S}} \mathbb{Q} : \mathrm{Sp} \to \mathrm{Sp}_{\mathbb{Q}}$ is left adjoint to the inclusion $\mathrm{Sp}_{\mathbb{Q}} \subseteq \mathrm{Sp}$

4. $\mathrm{Sp}_{\mathbb{Q}} \subseteq \mathrm{Sp}$ is closed under all limits and colimits.

5. $\mathrm{Sp}_{\mathbb{Q}} \cong \mathrm{Mod}_{H\mathbb{Q}} \simeq D(\mathbb{Q})$

6. $\mathrm{Sp}_{\mathbb{Q}}$ is symmetric monoidal with $X \otimes Y = X \otimes_{\mathbb{S}} Y = X \otimes_{H\mathbb{Q}} Y$ with tensor unit $H\mathbb{Q}$

Exercise 14.14. *Show this using that $H\mathbb{Q} \otimes_{\mathbb{S}} H\mathbb{Q} = H\mathbb{Q}$*

Slogan: Any spectrum X can be "recovered" from $X_p^\wedge$ for all p, $X_{\mathbb{Q}}$ and a certain gluing map.

Indeed we have maps $X \to X_p^\wedge$, $X \to X_{\mathbb{Q}}$ and $X_{\mathbb{Q}} \to (X_{\mathbb{Q}})_p^\wedge =: X_{\mathbb{Q}_p}$ which give a commutative square:

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^\wedge \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X)_{\mathbb{Q}} \end{array}$$

Called the Hasse- or fracture square.

Theorem 14.15. – *This square is a pullback for any spectrum X*

– *Moreover we have a pullback of ∞-categories:*

$$\begin{array}{ccc} \mathrm{Sp} & \xrightarrow{(-)^\wedge_p} & \prod_p \mathrm{Sp}^\wedge_p \\ \downarrow & & \downarrow \\ \mathrm{Sp}_{\mathbb{Q}}^{\Delta_1} & \xrightarrow[\mathrm{ev}_1]{} & \mathrm{Sp}_{\mathbb{Q}} \end{array} \qquad \begin{array}{c} (X_p) \\ \downarrow \\ \prod_p (X_p)_{\mathbb{Q}} \end{array}$$

For each $n = 0, 1, 2, \ldots, \infty$ have a pullback:

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}) & \xrightarrow{(-)^\wedge_p} & \prod_p \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}^\wedge_p) \\ \downarrow & & \downarrow \\ (\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}_{\mathbb{Q}}))^{\Delta_1} & \xrightarrow[\mathrm{ev}_1]{} & \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}_{\mathbb{Q}}) \end{array}$$

Proof sketch of the first part. It suffices to show that the map on horizontal fibers is an equivalence. This map is the rationalization hence it suffices to prove that the upper horizontal fiber is rational. This follows since the on the fiber of the map $X \to X^\wedge_p$ the prime p acts invertibly since the mod p reduction is 0. Combining this for al primes gives the claim. □

15 THH of the integers

Theorem 15.1. (Böchsktedt)
We have an isomorphism:

$$THH_*(Z) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}/n, & * = 2n-1 \\ 0, & \text{else} \end{cases}$$

In fact, $THH_*(\mathbb{Z})$ is the homology of the DGA:

$$THH_*(\mathbb{Z}) = H_*\left(\mathbb{Z}[x] \otimes \Lambda(e),\ |x| = 2,\ |e| = 1,\ \partial x = e,\ \partial e = 0\right)$$

These are equivalent, indeed the complex of the DGA has the form:

$$\ldots \xrightarrow{0} \mathbb{Z}x^2 \xrightarrow{\cdot 2} \mathbb{Z}ex \xrightarrow{0} \mathbb{Z}x \xrightarrow{\cdot 1} \mathbb{Z}e \xrightarrow{0} \mathbb{Z} \to 0$$

Remark 15.2. One can show that in fact, $THH(\mathbb{Z})$ is as an $\mathbb{E}_1$-algebra over $H\mathbb{Z}$ given as $H(\mathbb{Z}[x] \otimes \Lambda(e), \partial)$

As a consequence of these results we get that:

$$THH(\mathbb{Z})/p = THH(\mathbb{Z}) \otimes_{H\mathbb{Z}} H\,\mathbb{F}_p$$

is on homotopy groups isomorphic to the homology of the DGA:

$$\left(\mathbb{F}_p[x] \otimes_{\mathbb{F}_p} \Lambda(e), \partial x = e, \partial e = 0\right)$$

Exercise 15.3. *Compute the homology ring of this.*

Definition 15.4. For a ring (spectrum) R we define:

– $THH(R, \mathbb{Z}_p) = THH(R)^\wedge_p$

– $THH(R, \mathbb{Q}) = THH(R)_{\mathbb{Q}}$

– $THH(R, \mathbb{Q}_p) = THH(R, \mathbb{Z}_p)_{\mathbb{Q}}$

and hence we have the Hasse square:

$$\begin{array}{ccc} THH(R) & \longrightarrow & \prod_p THH(R, \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ THH(R, \mathbb{Q}) & \longrightarrow & \prod_p THH(R, \mathbb{Q}_p) \end{array}$$

Lemma 15.5. *We have:*

$$THH(R, \mathbb{Z}_p) = THH(R_p^\wedge, \mathbb{Z}_p) = HH(R_p^\wedge / \mathrm{Sp}_p^\wedge)$$

and moreover:

$$THH(R, \mathbb{Q}) = THH(R_\mathbb{Q}/H\mathbb{Q}) = HH(R_\mathbb{Q}/\mathbb{Q})$$

Proof. For the first part recalling how the colimits in $\mathrm{Sp}_p^\wedge$ are computed we immediately get:

$$\begin{aligned} THH(R, \mathbb{Z}_p) &= \left(\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{B}^{\mathrm{cyc}}(R) \right)_p^\wedge \\ &= \left(\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{B}^{\mathrm{cyc}}(R_p^\wedge) \right)_p^\wedge = THH(R_p^\wedge, \mathbb{Z}_p) \end{aligned}$$

Moreover the right hand equality follows by noting that $R_p^\wedge \otimes_\mathbb{S} R_p^\wedge \simeq R_p^\wedge \otimes_{\mathbb{S}_p^\wedge} R_p^\wedge$ and hence the Bar complexes agree.
For the second statement we see from our basechange formula:

$$THH(R, \mathbb{Q}) = THH(R) \otimes_\mathbb{S} H\mathbb{Q} \simeq THH(R \otimes_\mathbb{S} H\mathbb{Q}/H\mathbb{Q}) \simeq THH(R_\mathbb{Q}/H\mathbb{Q})$$

□

Example 15.6.

$$H(\mathbb{Z}, \mathbb{Q}) = HH(\mathbb{Q}/\mathbb{Q}) = H\mathbb{Q}$$

Theorem 15.7. *We have that:*

$$THH(\mathbb{Z}, \mathbb{Z}_p) \simeq \begin{cases} \mathbb{Z}_p, & * = 0 \\ \mathbb{Z}_p/n\mathbb{Z}_p, & * = 2n-1 \\ 0, & \text{else} \end{cases}$$

which is again isomorphic to the homology of the complex $(\mathbb{Z}_p \otimes \Lambda(e),\ \partial x = e,\ \partial e = 0)$

It is immediate that the first theorem implies this one but the converse is also true:

Indeed. From the second statement we can read off that $THH(\mathbb{Z}, \mathbb{Q}_p) \simeq H\mathbb{Q}_p$ and thus the fracture square looks like:

$$\begin{array}{ccc} THH(\mathbb{Z}) & \longrightarrow & \prod_p THH(\mathbb{Z}, \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ H\mathbb{Q} & \longrightarrow & \prod_p H\mathbb{Q}_p \end{array}$$

and hence from the long exact seuqnce we immediately get $THH_0(\mathbb{Z}) = \mathbb{Z}$ and more interestingly:

$$THH_n(\mathbb{Z}) = \prod_p THH_n(\mathbb{Z}, \mathbb{Z}_p) = \prod_p \mathbb{Z}_p/n\mathbb{Z}_p = \mathbb{Z}/n\mathbb{Z}$$

□

Definition 15.8. We define a commutative ring spectrum:

$$\mathbb{S}[z] = \mathbb{Z}[\mathbb{N}] = \Sigma^\infty_+ \mathbb{N}$$

Where $\mathbb{N}$ is considered as commutative algebra in $(\mathcal{S}, \times)$.

– To give a discrete commutative ring R the structure of an $\mathbb{S}[z]$-algebra is equivalent to giving an element $\pi \in R$ and mapping $z \mapsto \pi$ via the factorization:

$$\mathbb{S}[x] \to H\mathbb{Z}[x] \to HR$$

We consider $\mathbb{Z}$ as an $\mathbb{Z}[z]$-algebra by sending z to p

Warning: If instead of a discrete ring we consider a commutative rings spectrum R, then an $\mathbb{E}_\infty$-map $\mathbb{S}[z] \to R$ is not the same as an element $z \in \pi_0 R$, in other words $\mathbb{S}[z]$ is *not* the free $\mathbb{E}_\infty$-algebra on a single generator. (It is however a free $\mathbb{E}_1$-algebra)

Theorem 15.9. (Relative Böckstedt periodicity)
We have that:

$$THH_*(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p) \simeq \mathbb{Z}_p[x]$$

Proof. We have:

$$\begin{aligned} THH(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p)/p &\simeq THH(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p) \otimes_{H\mathbb{Z}} H\mathbb{F}_p \\ &\simeq THH(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p) \otimes_{\mathbb{S}[z]} \mathbb{S} \\ &\simeq THH(\mathbb{Z} \otimes_{\mathbb{S}[z]} \mathbb{S}, \mathbb{Z}_p) \\ &\simeq THH(\mathbb{F}_p) \end{aligned}$$

Now study the long exact sequence :

$$THH_{*+1}(\mathbb{F}_p) \longrightarrow THH_*(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p) \xrightarrow{\cdot p} THH_*(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p) \longrightarrow THH_*(\mathbb{F}_p)$$

Exercise 15.10. *Use this to deduce that $THH_*(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p)$ is concentrated in even degrees and p-torsion free.*

Hence there exists an element $x \in THH(+2(\mathbb{Z}/\mathbb{S}[z]), \mathbb{Z}_p)$ lifting the Böckstedt element $x \in THH_2(\mathbb{F}_p)$. Thus we get a map:

$$\mathbb{Z}_p[x \to THH_*(\mathbb{Z}/\mathbb{S}[z], \mathbb{Z}_p)]$$

which is an isomorphism after mod p reduction i.e. it is an isomoprhism. □

Now note that we have:

$$\begin{aligned} THH(R/\mathbb{S}[z]) &= THH(R) \otimes_{THH(\mathbb{S}[z])} \mathbb{S}[z] \\ &= THH(R) \otimes_{H\mathbb{Z} \otimes_{\mathbb{S}} THH(\mathbb{Z}[z])} (H\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S}[z]) \\ &\simeq THH(R) \otimes_{HH(\mathbb{Z}[z]/\mathbb{Z})} \mathbb{Z}[z] \end{aligned}$$

Using the HKR filtration we can deduce the following spectral sequence:

Proposition 15.11. *There is a strongly convergent, multiplicative first quadrant spectral sequence:*

$$THH_n(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}} \Lambda(dz) \simeq THH_n(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \Omega^m_{\mathbb{Z}[z]/\mathbb{Z}} \implies THH_{n+m}(R)$$

All of this works the same with $\mathbb{Z}_p$ coefficients. Now we can prove the theorem we started the lecture with:

Proof. The spectral sequence takes the form: □

16 The circle action on THH

Definition 16.1. For any grouplike $\mathbb{E}_1$-algebra G in $\mathcal{S}$ and an ∞-category $\mathcal{C}$ we define the category of objects of $\mathcal{C}$ with G-action as:

$$\mathrm{Rep}_G(\mathcal{C}) := \mathrm{Fun}(BG, \mathcal{C})$$

Given an action of G on $X \in \mathcal{C}$ we define:

$$X_{hG} := \operatorname*{colim}_{BG} X$$

$$X^{hG} := \lim_{BG} X$$

Exercise 16.2. *As a special case if X has the trivial action them we get:*

$$X_{hG} \simeq X \times BG$$

$$X^{hG} \simeq \mathrm{Map}(BG, X)$$

Proposition 16.3. *We have an equivalence:*

$$\mathrm{Fun}(BS^1, D(\mathbb{Z})) \simeq \mathrm{Mod}_A(D(\mathbb{Z}))$$

Where A was the DGA $\mathbb{Z}[\varepsilon]/\varepsilon^2$ with the 0 differential.

Proof Sketch. One can show that:

$$\mathrm{Fun}(BS^1, D(\mathbb{Z})) \simeq \mathrm{Mod}_{C_*(S^1)}$$

moreover we have maps $\mathrm{Free}_{\mathbb{E}_1}(\varepsilon) \to C_*(S^1)$ and $\mathrm{Free}_{\mathbb{E}_1}(\varepsilon) \to A$ which factor trough equivalences from $\tau_{\leq 1}\mathrm{Free}_{\mathbb{E}_1}(\varepsilon)$ which are multiplicative since $\tau_{\leq}1$ is lax monoidal on $D(\mathbb{Z})_{\geq 0}$ □

Remark 16.4. This equivalence is *not* compatible with the symmetric monoidal structures.

Theorem 16.5.
On $HH(R/\mathcal{C})$ we have a natural S^1-action, i.e. a refinement:

$$HH : \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \to \mathrm{Fun}(BS^1, \mathcal{C})$$

which agrees with the S^1-action obtained from the Connes-operator.

Definition 16.6. The *paracyclic category* Λ_∞ is the 1 category with:

- Objects: Totally ordered sets with $\mathbb{Z}$-action, equivalent to $\frac{1}{n}\mathbb{Z}$
- Morphisms: Equivariant, order preserving maps

Remark 16.7. We have an $S^1 \simeq B\mathbb{Z}$-action on Λ_∞ which is given by the map of 1-categories:

$$B\mathbb{Z} \times \Lambda_\infty \to \Lambda_\infty$$

which is the projection on points and on Hom-sets is given by:

$$\mathbb{Z} \times \mathrm{Hom}_{\Lambda_\infty}(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z}) \to \mathrm{Hom}_{\Lambda_\infty}(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z})$$
$$(n, \phi) \mapsto \phi + n$$

Definition 16.8. The *cyclic* category Λ is the 1-category with:

- Objects: Same as Λ_∞

– Morphisms given by the qoutient: $\mathrm{Hom}_\Lambda(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z}) = \mathrm{Hom}_{\Lambda_\infty}(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z})_\mathbb{Z}$

Lemma 16.9. *On the level of ∞-categories we have that:*

$$N(\Lambda) \simeq N(\Lambda_\infty)_{hS^1}$$

Lemma 16.10. *We have an equivalence:*

$$\mathrm{Fun}(N(\Lambda), \mathcal{C}) \simeq \mathrm{Fun}(N(\Lambda_\infty), \mathcal{C})^{hS^1}$$

Proof. This is almost the universal property except that the functor category is not a groupoid. However doing the usual trick gives:

$$\begin{aligned} \mathrm{Map}(\mathcal{D}, \mathrm{Fun}(N(\Lambda), \mathcal{C})) &\simeq \mathrm{Map}(N(\Lambda), \mathrm{Fun}(\mathcal{D}, \mathcal{C})) \\ &\simeq \mathrm{Map}(N(\Lambda_\infty), \mathrm{Fun}(\mathcal{D}, \mathcal{C}))^{hS^1} \\ &\simeq \mathrm{Map}(\mathcal{D}, \mathrm{Fun}(N(\Lambda_\infty), \mathcal{C}))^{hS^1} \end{aligned}$$

□

Definition 16.11. A cyclic object in $\mathcal{C}$ is a functor $N(\Lambda^{\mathrm{op}}) \to \mathcal{C}$.
We have a functor:

$$\begin{aligned} \Delta &\to \Lambda_\infty \\ [n-1] &\mapsto \mathbb{Z} \times [n-1] \quad \text{(with lexicographic order)} \end{aligned}$$

where the $\mathbb{Z}$ action is induced by the natural one on $\mathbb{Z}$. With our previous notation $\mathbb{Z} \times [n-1] \cong \frac{1}{n}\mathbb{Z}$. The *underlying simplicial object* is obtained via the map:

$$\mathrm{Fun}(N(\Lambda^{\mathrm{op}}), \mathcal{C}) \to \mathrm{Fun}(N(\Lambda_\infty^{\mathrm{op}}), \mathcal{C}) \to \mathrm{Fun}(N(\Delta^{\mathrm{op}}), \mathcal{C})$$

Lemma 16.12. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Fun}(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}) & \xrightarrow{\mathrm{colim}} & \mathcal{C} \\ \downarrow{\scriptstyle \mathrm{res}} & \nearrow{\scriptstyle \mathrm{colim}} & \\ \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) & & \end{array}$$

Lemma 16.13. *For a cyclic object X the colimit* $\mathrm{colim}_{\Lambda_\infty^{\mathrm{op}}} X \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} X$ *carries a natural S^1-action.*

Proof.

$$\mathrm{Fun}(N(\Lambda^{\mathrm{op}}), \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(N(\Lambda_\infty^{\mathrm{op}}), \mathcal{C})^{hS^1} \xrightarrow{(\mathrm{colim})^{hS^1}} \mathcal{C}^{hS^1} \simeq \mathrm{Fun}(BG, \mathcal{C})$$

□

Proof of the Theorem. Denote $\mathrm{Cut}^\mathbb{Z} : \Lambda_\infty^{\mathrm{op}} \to \mathrm{Ass}_{\mathrm{act}}^\otimes$ the functor that maps S to the set of "$\mathbb{Z}$-equivariant cuts" i.e. cuts of $S_\mathbb{Z}$ with preimage of $(S_i) \in \mathrm{Cut}^\mathbb{Z}(S)$ in $\mathrm{Cut}^\mathbb{Z}(T)$ under some map $f : S \to T$ consist of all cuts "between" $f(S_0), f(S_1)$. This functor is $\mathbb{Z}$-invariant on Morphisms so it factors through $\Lambda \to \mathrm{Ass}_{\mathrm{act}}^\otimes$. The restriction to Δ^{op} is precisely $\mathrm{Cat}^{\mathrm{cyc}} : \Delta^{\mathrm{op}} \to \mathrm{Ass}_{\mathrm{act}}^\otimes$ □

Exercise 16.14. *Show the last statement.*

17 Negative Topological Cyclic Homology

Definition 17.1. For a rings spectrum R we define the *negative topological cyclic Homology* of R as:

$$TC^-(R) := THH(R)^{hS^1}$$

Definition 17.2. And ∞-category $\mathcal{C}$ is called *stable* if it has finite limits and colimits and any square in $\mathcal{C}$:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout iff it is a pullback.

Lemma 17.3. *If $\mathcal{C}$ is stable, then there is a unique lift:*

$$\begin{array}{ccc} & & \mathrm{Sp} \\ & \overset{\mathrm{map}}{\nearrow} & \downarrow \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow[\mathrm{Map}]{} & \mathcal{S} \end{array}$$

which is exact in both variables separately.

Lemma 17.4. *For I a small ∞-category we have that:*

1. $\mathrm{Fun}(I, \mathrm{Sp})$ *is stable*
2. $\lim_I F \simeq \mathrm{map}_{\mathrm{Fun}(I,\mathrm{Sp})}(\mathrm{const}_{\mathbb{S}}, F)$

Exercise 17.5. *Prove this.*

Proposition 17.6. *We have that:*

$$HC^-(R) \simeq HH(R)^{hS^1} \in D(Z)$$

Proof. Identify $D(\mathbb{Z})$ with $\mathrm{Mod}_{H\mathbb{Z}}$, then:

$$\begin{aligned} HH(R)^{hS^1} &\simeq \mathrm{map}_{\mathrm{Fun}(BS^1,\mathrm{Sp})}(\mathbb{S}, HH(R)) \\ &\simeq \mathrm{map}_{\mathrm{Fun}(BS^1,\mathrm{Mod}_{HZ})}(H\mathbb{Z}, HH(R)) \\ &\simeq \mathrm{map}_{\mathrm{Fun}(BS^1,D(\mathbb{Z}))}(\mathbb{Z}, HH(R)) \\ &\simeq \mathrm{map}_{\mathrm{Mod}(C_*(S^1))}(\mathbb{Z}, HH(R)) \\ &\simeq \mathrm{map}_{\mathrm{Mod}(A)}(\mathbb{Z}, HH(R)) \\ &\simeq \mathrm{RHom}_A(\mathbb{Z}, HH(R)) := HC^-(R) \end{aligned}$$

□

Lemma 17.7. *The functor:*

$$(-)^{hS^1} : \mathrm{Fun}(BS^1, \mathrm{Sp}) \to \mathrm{Sp}$$

is lax symmetric monoidal. Moreover the S^1-action on $THH(R)$ (and $HH(R)$) is compatible with the ring structure obtained if R is $\mathbb{E}_\infty$. Hence in this case we get an $\mathbb{E}_\infty$-structure on $TC^-(R)$ and $HC^-(R)$.

Lemma 17.8. *Let $BS^1 \to \mathrm{Sp}$ be an S^1-action on HA, then we have:*

$$\pi_*(HA)^{hS^!} \cong A[t] = A \otimes \mathbb{Z}[t], \quad |A| = -2$$

Proof. The full subcategory of Sp on all Eilenberg-MacLane spectra (in degree 0) is a 1 category, since:

$$\mathrm{Map}(HA, HB) \simeq \mathrm{Hom}(A, B)$$

so a functor $BS^1 \to \mathrm{Sp}$ sending $* \mapsto HA$ is constant since BS^1 is 1-connected. Using this we get that:

$$HA^{hS^1} \simeq \mathrm{map}(\Sigma^\infty BS^1, HA)$$

Exercise 17.9. *Show this by writing the suspension spectrum as a limit.*

which has homotopy groups given by $H^*(BS^1, A) = H^*(\mathbb{C}P^\infty, A)$ □

Theorem 17.10. (Homotopy Fixed Point Spectra Sequence [HFPSS]) *There is a multiplicative, conditionally convergent spectral sequence:*

$$\pi_*(X)[t] \implies \pi_*\left(X^{hS^1}\right), \quad |t| = (-2, 0)$$

Proof sketch. Have the Whitehead tower:

$$\begin{array}{ccccccc}
\vdots & & & & \vdots & & \\
\downarrow & & & & \downarrow & & \\
\tau_{\geq 2}X & \xrightarrow{\text{cofib}} & H\pi_1 X[1] & & (\tau_{\geq 2}X)^{hS^1} & \longrightarrow & (H\pi_1 X)^{hS^1}[1] \\
\downarrow & & & \overset{(-)^{hS^1}}{\rightsquigarrow} & \downarrow & & \\
\tau_{\geq 1}X & \xrightarrow{\text{cofib}} & H\pi_0 X & & (\tau_{\geq 1}X)^{hS^1} & \longrightarrow & (H\pi_0 X)^{hS^1} \\
\downarrow & & & & \downarrow & & \\
\tau_{\geq 0}X & & & & (\tau_{\geq 0}X)^{hS^1} & &
\end{array}$$

□

Remark 17.11. An S^1-action on X actually gives a map $\Sigma^\infty_+ S^1 \otimes X \to X$. The canonical map $\Sigma^\infty_+ S^1 \to \Sigma^\infty_+ * \simeq \mathbb{S}$ splits, so we get an equivalence

$$\Sigma^\infty_+ S^1 \simeq \mathbb{S} \oplus \Sigma^\infty S^1 \simeq \mathbb{S}^0 \oplus \mathbb{S}^1$$

(for the first time we really mean the reduced suspension)
Thus we get a map $\Sigma X \simeq \mathbb{S}^1 \otimes X \to X$ i.e. a map:

$$b : \pi_n(X) \to \pi_{n+1}(X)$$

Lemma 17.12. *In the HFPSS the differential d_2 is determined by:*

$$d_2\alpha = b\alpha \cdot t$$

$$d_2 t = \eta t^2$$

for $\alpha \in \pi_*(X)$

Remark 17.13.

1. Observe that $THH_*[\mathbb{F}_p][t] = \mathbb{F}_p[x][t]$ is concentrated in even degrees and hence all differentials of the HFPSS for $TC^-(\mathbb{F}_p)$ are zero and so the E_∞-page is given by $\mathbb{F}_p[x][t]$.

2. Hence we have that $TC^-_{2k+1}(\mathbb{F}_p) = 0$ and moreover $TC^-_{2k}(\mathbb{F}_p)$ are complete filtered abelian groups with associated graded given by a sequence of $\mathbb{F}_p$'s. Now choose $\tilde{t} \in TC^-_{-2}$ and $\tilde{x}i \in TC^-_2$ lifts of t, x.

3. $TC_0^-(\mathbb{F}_p) \to TC_0^-(\mathbb{F}_p) \to \mathbb{F}_p$ is surjective with kernel given by the ideal generated by $\tilde{t}\tilde{x}$. So there is a relation $p = a\tilde{t}\tilde{x}$

4. We have a map $THH(\mathbb{F}_p) \to HH(\mathbb{F}_p)$ which is an iso on $\tau_{\leq 2}$, so we get an equivalence:

$$(\tau_{\leq 2}THH(\mathbb{F}_p))^{hS^1} \xrightarrow{\sim} (\tau_{\leq 2}HH(\mathbb{F}_p))^{hS^1}$$

We saw in the SS for HC^- that $\tilde{t}\tilde{x} = p$ for suitable choice of lifts. Hence we see that in fact a has degree 0, since else p would die in the 2-truncation and hence a is a unit.

5. We can modify our choice of $\tilde{x}$ such that $a = 1$ so that $\tilde{t}\tilde{x} = 0$. Hence we get a map:

$$\mathbb{Z}_p[\tilde{t}, \tilde{x}]/(p - \tilde{t}\tilde{x}) \to TC_*^-(\mathbb{F}_p)$$

Theorem 17.14. *The map above gives an isomorphism:*

$$TC_*^-(\mathbb{F}_p) \xrightarrow{\sim} \mathbb{Z}_p[\tilde{t}, \tilde{x}]/(p - \tilde{t}\tilde{x})$$

i.e. we have:

$$TC_{2k}^-(\mathbb{F}_p) \simeq \mathbb{Z}_p \text{ gen. by } \begin{cases} \tilde{x}^k, & \text{if } k \geq 0 \\ \tilde{t}^{-k} & \text{if } k \leq 0 \end{cases}$$

18 The Tate Construction

Let G be a group object in $(\mathcal{S}, \times)$ i.e. a group-like $\mathbb{E}_1$-space. Have the delooping $BG \in \mathcal{S}$, then of any ∞-category $\mathcal{C}$ we have the category of objects in $\mathcal{C}$ with G-action:

$$\mathcal{C}^{BG} := \mathrm{Fun}(BG, \mathcal{C})$$

Construction:
Consider the underlying space of G as an object of $\mathcal{S}^{G\times G}$ by letting $G \times G$ act on G as:

$$(g, h) \cdot x = gxh^{-1}$$

Definition 18.1. We define a spectrum $D_G \in \mathrm{Sp}^{BG}$ called the *dualizing spectrum* of G. as:

$$D_G := (\Sigma_+^\infty G)^{h(G\times 1)}$$

with its remaining $G = 1 \times G$-action

Example 18.2. Assume that G is finite, then:

$$\Sigma_+^\infty G = (\bigoplus_{g\in G} \mathbb{S})^{hG} \simeq \mathbb{S}$$

Proof. Have the HFSS:

$$H^*(G, \oplus_{g\in G}\pi_*(\mathbb{S})) \implies \pi_*((\oplus_{g\in G}\mathbb{S})^{hG})$$

It is a classical result that:

$$H^*(G, \oplus_{g\in G}A) = \begin{cases} A, & * > 0 \\ 0, & * = 0 \end{cases}$$

Hence the spectral sequence degenerates and gives the result. □

Hence for a finite group we have $D_G = \mathbb{S}^{\mathrm{triv}}$.
If G is a compact Lie Group we have the following theorem:

Theorem 18.3. *We have that $D_G = \mathbb{S}^{\mathfrak{g}}$ where for any vector space V we define $\mathbb{S}^V = \Sigma_+^\infty(V^+)$ and G acts on $\mathfrak{g}$ by the adjoint representation.*

Example 18.4. Consider the circle group $\mathbb{T} = U(1) = S^1$, then we get:

$$D_{\mathbb{T}} = (\mathbb{S}^1)^{\mathrm{triv}}$$

Assume that $BG \simeq (M, m_0)$ where M is a closed smooth manifold, i.e. any closed manifold (M, m_0) and $G = \Omega M$

Theorem 18.5. *We have $D_G = \mathbb{S}^{-T_{m_0}M}$ which as a functor is given by:*

$$BG \simeq M \to \mathrm{Sp}$$

$$m \mapsto \mathbb{S}^{-T_m M}$$

which is the straightening of the tangent bundle.

Remark 18.6. For any space X one can define a dualizing spectrum:

$$D_X =: X \to \mathrm{Sp}$$

for example by defining this functor on connected components: $X \simeq \coprod BG_i$ and on BG_i as before.

Construction:
Let $\mathcal{C}$ be a stable ∞-category which has all limits and colimits. For any $E \in \mathrm{Sp}$ and $X \in \mathcal{C}$ there is an object:

$$E \otimes X \in \mathcal{C}$$

($\mathcal{C}$ is a module over Sp) defined such that:

$$- \otimes X : \mathrm{Sp} \to \mathcal{C}$$

sends colimits of spectra to colimits in $\mathcal{C}$ and extends the tensoring over $\mathcal{S}$. This can also be characterized by saying that it is adjoint to the mapping spectrum.

Definition 18.7. G a group object in $\mathcal{S}$ and $X \in \mathcal{C}^{BG}$. We define the *norm map*:

$$N_G : (DG \otimes_{\mathbb{S}} X)_{hG} \to X^{hG}$$

as the composite:

$$\left((\Sigma^\infty_+ G)^{h(G\times 1)} \otimes_{\mathbb{S}} x\right)_{h(1\times G)} \to \left((\Sigma^\infty_+ G \otimes_{\mathbb{S}} X)^{h(G\times 1)}\right)_{h(1\times G)} \to \left((\Sigma^\infty_+ G \otimes_{\mathbb{S}} X)_{h(1\times G)}\right)^{h(G\times 1)} \simeq X^{hG}$$

where we consider X as a $G\times G$-spectrum where $G\times 1$ acts trivially and $1\times G$ is the given G-action on X.

Exercise 18.8. *Show the last equivalence.*

Example 18.9. 1. If G is finite then:

$$(D_G \otimes X)_{hG} \simeq X_{hG} \to X^{hG}$$

For $X = \mathrm{Sp}$, $X = HM$, M and abelian group with G-action. Then we have a map:

$$\begin{array}{ccc} HM_{hG} & \longrightarrow & HM^{hG} \\ \downarrow & & \uparrow \\ H(M_G) & \longrightarrow & H(M^G) \end{array}$$

where the lower map is the classical norm map. This follows from the fact that for general X the composite:

$$X \to X_{hG} \xrightarrow{N_G} X^{hG} \to X$$

is given by the sum of the multiplication by g maps $\rho_g : X \to X$

Exercise 18.10. *Show this!*

2. For $G = \mathbb{T}$ we get a map;

$$\Sigma X_{h\mathbb{T}} = (D_{\mathbb{T}} \otimes X)_{h\mathbb{T}} \to X^{h\mathbb{T}}$$

Theorem 18.11. *The norm map $(D_G \otimes X)_{hG} \to X^{hG}$ is an equivalence provided that one of the following conditions hold:*

1. *BG is a finite CW-complex*
2. *X is induced, that is $X \simeq \Sigma^\infty_+ \otimes Y$ where G acts only on Σ^∞_+*

Proof. 1. In this case all limits in the interchange maps are finite.

2. In the induced case the colimits and limits are still close enough to finite (Statement about boundedness of group (co-)homology)

□

Example 18.12. Let $BG = M$ be a closed manifold $\mathcal{C} = \mathrm{Sp}$, $X = H\mathbb{Z}^{\mathrm{triv}}$

$$(H\mathbb{Z}[-n])_{hG} = (\mathbb{S}^{-TM} \otimes H\mathbb{Z})_{hG} = (D_G \otimes X)_{hG} \xrightarrow{\sim} X^{hG} = H\mathbb{Z}^{Bg} = \mathrm{map}(BG, X)$$

which on homotopy groups induces an isomorphism:

$$H_{*+n}(M, \mathbb{Z}^{\mathrm{orient}}) \xrightarrow{\sim} H^{-*}(M, \mathbb{Z})$$

which is precise Poincaré duality!
Replacing $H\mathbb{Z}$ by any spectrum we get Poincaré duality for ordinary (co-)homology theories.
If BG is a finite CW-comples this gives a a generalized version of Poincaré duality:

$$H_*(X, D_X) \to H^{-*}(X)$$

D_X is a parametrized sphere i.e. has underlying spectrum $\mathbb{S}^n$ iff X is a Poincaré duality space.

Theorem 18.13. *The transformation:*

$$(D_G \otimes -)_{hG} \to (-)^{hG}$$

exhibits the functor $(D_G \otimes -)_{hG}$ as the universal functor over $(-)^{hG}$ which preserves colimits, i.e. the assembly map. In fact this uniquely determines D_G

Definition 18.14. For $X \in \mathcal{C}^{BG}$ we define the *Tate Construction* as:

$$X^{tG} := \mathrm{cofib}(N_G)$$

Example 18.15. 1. If BG is finite then $X^{tG} = 0$ for all X.

2. For G finite and $X = HM$, M an abelian grou with action we have:

$$(HM)^{tG} = \mathrm{cofib}(N_G : HM_{hG} \to HM^{hG})$$

The homotopy groups are given by the classical *Tate cohomology*:

$$\pi_*((HM)^{tg}) = \widehat{H}^{-*}(G, M)$$

Exercise 18.16. *Describe these homotopy groups in terms of group (co)-homology.*

Theorem 18.17. *Assume that $\mathcal{C}$ is a symmetric monoidal ∞-category such that the tensor product commutes with colimits in both variables separately. Then the functor $(-)^{tG} : \mathcal{C}^{BG} \to \mathcal{C}$ admits a (unique) lax symmetric monidal structure such that:*

$$(-)^{hG} \to (-)^{tG}$$

admits a refinement to a symmetric monoidal transformation.

Corollary 18.18. *If $A \in \mathrm{CAlg}(\mathcal{C}^{BG}) \simeq \mathrm{CAlg}(\mathcal{C})^{BG}$ then $A^{hG} \to A^{tG}$ is a map of commutative algebras.*

19 Topological Periodic Homology

Definition 19.1. For a ring R we define the *topological periodic homology* of R as:

$$TP(R) := THH(R)^{tS^1}$$

Remark 19.2. Since $(-)^{tG}$ is lax symmetric symmetric monoidal in a way that is compatible with $(-)^{hG}$ we have that for a commutative ring R the spectrum $TP(R)$ is a $TC^-(R)$-algebra.

Proposition 19.3. *For spectra X with G-action there is a multiplicative, conditionally convergent spectra sequence:*

$$\pi_p\left((H\pi_q(X))^{tG}\right) \implies \pi_{p+q}(X^{tG})$$

Proof Sketch. Take the Whitehead filtration $\tau_{\geq \bullet}X$ on X. Applying $(-)^{tG}$ gives a filtered spectrum whose associated graded is given by applying $(-)^{tG}$ to Eilenberg MacLane Spectra which is where the LHS comes from. □

To use this result we need to know more about the Tate construction of Eilenberg MacLane Spectra. For $H\mathbb{Z}$, $G = S^1$ consier the cofiber sequence:

$$(S^{\mathrm{ad}_G} \otimes H\mathbb{Z})_{hG} \to H\mathbb{Z}^{hS^1} \to H\mathbb{Z}^{tS^1}$$

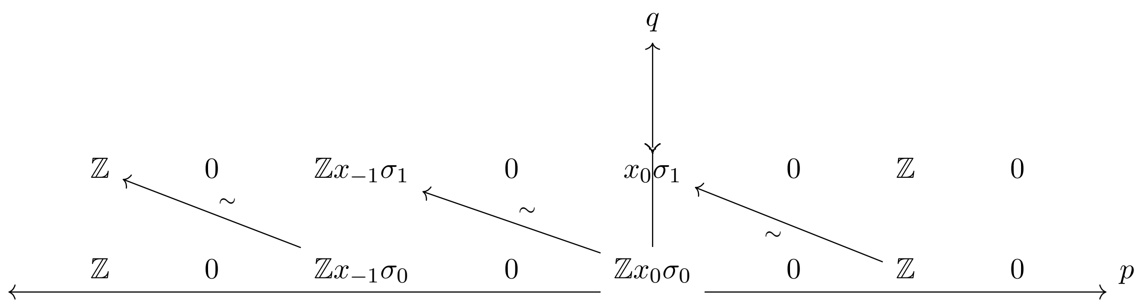
Now since G is abelian the adjoint representation is trivial. Moreover since it is 1-dimensional the representation sphere is given by S^1, hence the left hand term is $\Sigma H\mathbb{Z}_{hG}$. Since the fixed points have homotopy groups in non-positive degreese and the shift of the orbits only in positive degrees we get:

$$\pi_*(H\mathbb{Z}^{tS^1}) = \begin{cases} \mathbb{Z} & \text{for } * \text{ even} \\ 0 & \text{else} \end{cases}$$

Lemma 19.4. *We have that as an algebra over $H\mathbb{Z}^{hS^1}$:*

$$\pi_*(H\mathbb{Z}^{tS^1}) \simeq \mathbb{Z}[t^{\pm}]$$

Proof. We use that the Tate-construction annihilates induced representations and work "backwards" spectral sequence: Consider the Tate SS for $H\mathbb{Z}\otimes\Sigma^\infty_+ S^1$. We know that $(H\mathbb{Z}\otimes\Sigma^\infty_+ S^1)^1 = 0$ Fix generators x_i of $\pi_{2i}H\mathbb{Z}^{tS^1}$, σ_0, σ_1 of $H_*(S^1)$ where we set $x_{-i} = t^i$. Hence in the SS the generator in degree (i, j) is $x_i\sigma_j$.



Then we have that:

$$d_2(\sigma_0) = \pm t\sigma_1$$

so since the spectral sequence computing the homotopy fixed points is degnerate we get from the Leibniz rule that: $d_2(x_i\sigma_0) = \pm x_i \cdot t\sigma_1$. But since this needs to be a generator it is also given by $\pm x_{i-1}\sigma_1$ so we have:

$$x_i \cdot t = \pm x_{i-1}$$

Now choose the generators x_i such that all signs are + so we get that:

$$\pi_{2i} H\mathbb{Z}^{tS^1} = \mathbb{Z}[t^{\pm}]$$

□

Exercise 19.5. *Using a similar approach compute $\pi_*(H\mathbb{Z}^{tC_m})$ as a ring. (Hint: Use a free action of C_m on a space with easy homology groups.)*

Remark 19.6. Doing the same argument with $HA \otimes \Sigma^\infty_+ S^1$ gives that $\pi_* A^{tS^1} = A[t^{\pm}]$ as a $\mathbb{Z}$-module.

Proposition 19.7. *If X is an object of $\mathrm{Fun}(BS^1, \mathrm{Mod}_{H\mathbb{Z}})$ i.e. a module over $H\mathbb{Z}^{\mathrm{triv}}$ in $\mathrm{Fun}(BS^1, \mathrm{Sp})$ (The point is that the action is required to be $H\mathbb{Z}$-linear). In this case we have:*

$$X^{tS^1} \simeq X^{hS^1} \otimes_{H\mathbb{Z}^{hS^1}} H\mathbb{Z}^{tS^1}$$

and in particular:

$$\pi_* X^{tS^1} \simeq \pi_* X^{hS^1}[t^{-1}]$$

Proof Sketch. We have:

$$H\mathbb{Z}^{tS^1} \simeq \mathrm{colim}\left(H\mathbb{Z}^{hS^1} \xrightarrow{\cdot t} \Sigma^2 H\mathbb{Z}^{hS^1} \xrightarrow{\cdot t} \Sigma^4 H\mathbb{Z}^{hS^1} \xrightarrow{\cdot t} \dots\right)$$

as $H\mathbb{Z}^{hS^1}$-modules.

Exercise 19.8. *Prove this!*

So we need to show that:

$$X^{tS^1} = \mathrm{colim}(X^{hS^1} \xrightarrow{\cdot t} X^{hS^1} \to \dots)$$

- Works for X Eilenberg-MacLane
- Both sides are compatible with fiber sequences.
- To extend to *all* X we need some connectivity arguments.

□

Proposition 19.9. *We have that:*

$$HP(R) \simeq HH(R)^{tS^1}$$

Proof. Since $HH(R)$ is equivariantly an $H\mathbb{Z}$-module, $HH(R)^{tS^1}$ is obtained from $HH(R)^{tS^1} = HC(R)$ by inverting t and this was precisely how we defined HP. □

Proposition 19.10. *The Tate spectral sequence for tS^1 takes the form:*

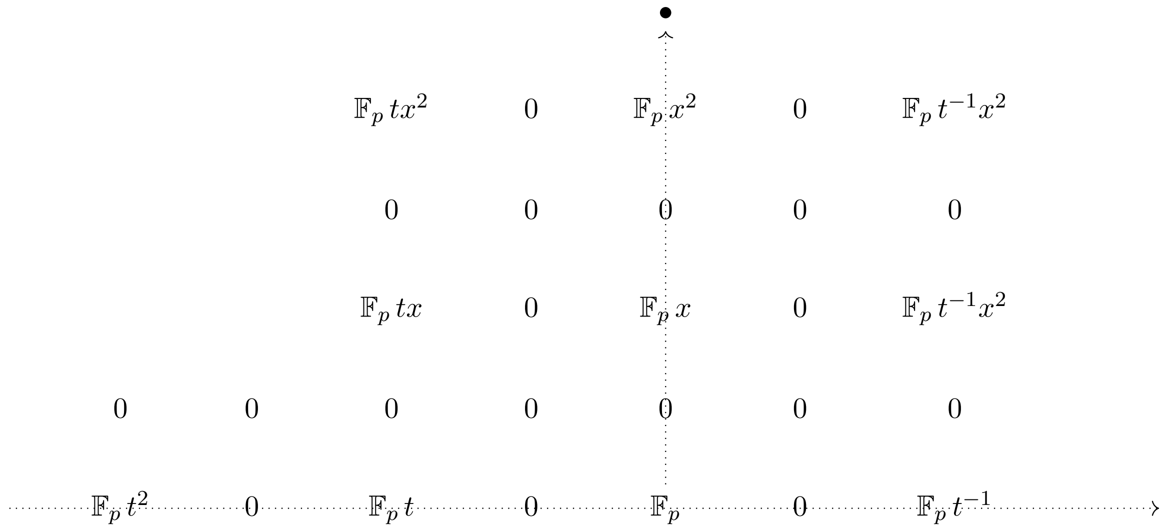
$$\pi_*(X)[t^{\pm t}] \implies \pi_*(X^{tS^1})$$

$$THH_*(R)[t^{\pm 1}] \implies TP_*(R)$$

Theorem 19.11. *We have:*

$$\pi_*(TP(\mathbb{F}_p)) \simeq \mathbb{Z}_p[t^{\pm 1}]$$

Proof. The Tate Spectral sequence is a peridocized version of the HFPSS and hence degenerates. It takes the form:



and receivesa map fprm the HFPSS which is an iso on the left quadrant. Hence in negative degrees we have:

$$\pi_{-2k}TP(\mathbb{F}_p) = \pi_{-2k}TC^-(\mathbb{F}_p) \cong \mathbb{Z}_p \cdot \tilde{t}$$

We can also choose a representative $\widetilde{t^{-1}} \in \pi_2 TP(\mathbb{F}_p)$ of t^{-1}. Then in fact by the ring structure:

$$\widetilde{t} \cdot \widetilde{t^{-1}} = 1 \quad \text{mod higher filtration}$$

and hence it is a unit so $\widetilde{t}$ is invertible and the negative powers $\widetilde{t}^{-k}$ generate the $\pi_{2k}(TP(\mathbb{F}_p) \cong \mathbb{Z}_p[\widetilde{t}^{\pm}]$ □

20 The Tate Diagonal

For abelian groups the map of sets $A \to A \otimes A,\ x \mapsto x \otimes x$ is *not* a homomorphism since we get "error terms" of the form $x \otimes y + y \otimes x$. Recall the norm map:

$$\begin{aligned} N : (A^{\otimes p})_{C_p} &\to (A^{\otimes p})^{C_p} \\ x_1 \otimes \cdots \otimes x_p &\mapsto \sum_{\sigma \in C_p} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)} \end{aligned}$$

Observation: The "diagonal" $A \to A^{\otimes p},\ x \mapsto x \otimes \cdots \otimes x$ induces a homomorphism:

$$A \to (A^{\otimes p})^{C_p} / N(A^{\otimes p}_{C_p})$$

This homomorphism exhibits $(A^{\otimes p})^{C_p}/N(A^{\otimes p}_{C_p})$ as A/p i.e. it is surjective with kernel pA.

Exercise 20.1. *Show these statements. For the second part show first that the functor taking A to the right hand side is additive and then deduce the claim by showing it for finitely generated abelian groups.*

Theorem 20.2. *There is a unique lax symmetric monoidal natural transformation:*

$$X \to (X^{\otimes p})^{tC_p}$$

of functors Sp $\to$ Sp

Lemma 20.3. *The Singer construction:*

$$X \mapsto (X^{\otimes p})^{tC_p}$$

is an exact functor.

Proof. First look at $Y \simeq X \oplus Z$, then we have:

$$(X+\mathbb{Z})^{\otimes p} \simeq X^{\otimes p} \oplus \mathbb{Z}^{\otimes p} \oplus \bigoplus_{p} (X^{\otimes p-1} \otimes Z) \oplus \bigoplus_{\binom{p}{2}} \left(X^{\otimes (p-2)} \otimes Z^{\otimes 2}\right) \oplus \dots$$

now note that the C_p-action on the error terms is the free action that permutes the summands (using for example that the binomial coefficients are divisible by p), hence they are induced and vanish after applying $(-)^{tC_p}$. In the general case think of a cofiber sequence:

$$X \to Y \to Z$$

as "2-stage filration" of Y:

$$\dots \to 0 \to X \to Y$$

with associated graded $Z \oplus X$. We also get a filtration on $Y^{\otimes p}$:

$$X^{\otimes p} \longrightarrow \bigoplus_p X^{\otimes (p-1)} \otimes Y \longrightarrow \cdots \longrightarrow \bigoplus_p X \otimes Y^{\otimes (p-1)} \longrightarrow Y^{\otimes p}$$

Which is precisely the Day convolution i.e. the symmetric monoidal structure on filtered objects. This is multiplicative on the associated graded which then becomes:

$$X^{\otimes p} \qquad \bigoplus_p X^{\otimes (p-1)} \otimes Z \qquad \cdots \qquad \bigoplus_p X \otimes Z^{\otimes (p-1)} \qquad Z^{\otimes p}$$

So after applying $(-)^{tC_p}$ we are left with:

$$(X^{\otimes p})^{tC_p} \qquad 0 \qquad \cdots \qquad 0 \qquad (Z^{\otimes p})^{tC_p}$$

This is now the associated graded of a filtration of $(Y^{\otimes p})^{tC_p}$ so we get a cofiber sequence:

$$(X^{\otimes p})^{tC_p} \to (Y^{\otimes p})^{tC_p} \to (Z^{\otimes p})^{tC_p}$$

which was the claim. □

Remark 20.4. Note that while $(\Sigma X)^{\otimes p} \simeq \Sigma^n (X^{\otimes p})$ we just showed:

$$((\Sigma X)^{\otimes p})^{tC_p} \simeq \Sigma (X^{\otimes p})^{tC_p}$$

Indeed, $(\Sigma X)^{\otimes p} \simeq \mathbb{S}^p \otimes sX^{\otimes p}$, but the C_p-astion involves a nontrivial action on $\mathbb{S}^p$ (Called the regular representation sphere). In fact $(\mathbb{S}^p)^{tC_p} \simeq (\mathbb{S}^1)^{tC_p}$

Lemma 20.5. (Stable Yoneda) *Let $\mathcal{C}$ be a stable ∞-category, we have a natural equivalence:*

$$\mathrm{map}_{\mathrm{Fun}^{\mathrm{ex}}}(\mathrm{map}(X,-), F) \simeq F(X)$$

Proof Sketch. A natural transformation $\mathrm{map}(X,-) \to F$ is the same as a sequence of compatible natural transformation:

$$\Omega^\infty \mathrm{Map}(\Sigma^{-n} X, -) \to \Omega^\infty \Sigma^n F(-)$$

i.e. by the ordinary Yoneda this is a sequence of points in :

$$\Omega^\infty \Sigma^n F(\Sigma^{-n} X) \simeq \Omega 6 \infty F(X)$$

i.e. we get that:

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathrm{Sp})}(\mathrm{map}(X,-), F) \simeq \Omega^\infty F(X)$$

□

Remark 20.6. In particular we have seen that natural transformations $X \to (X^{\otimes p})^{tC_p}$ correspond to maps $\mathbb{S} \to (\mathbb{S}^{\otimes p})^{tC_p} \simeq \mathbb{S}^{tC_p}$. Thus the Tate diagonal is also given by the composition:

$$\mathbb{S} \to \mathbb{S}^{hC_p} \xrightarrow{\text{can}} \mathbb{S}^{tC_p}$$

Lemma 20.7. *The functor* $\mathrm{map}(\mathbb{S}, -)$ *is initial among lax symmetric monoidal exact functors from* Sp *to itself.*

The following is a very deep theorem in stable homotopy theory:

Theorem 20.8. *For* $X \in \mathrm{Sp}$ *bounded below, then:*

$$X \to (X^{\otimes p})^{tC_p}$$

exhibits $(X^{\otimes p})^{tC_p}$ *as* $X_p^\wedge$ *(!!!). In particular* $(X^{\otimes p})^{tC_p}$ *has the same connectivity (!) and for the sphere we get:*

$$\mathbb{S}^{tC_p} \simeq \mathbb{S}_p^\wedge$$

Remark 20.9. The last equivalence is a special case of the Segal Conjecture (Carlson)

Definition 20.10. For an $\mathbb{E}_\infty$-ring R the multiplication map gives the *Tate-valued Frobenius*:

$$R \xrightarrow{\Delta} (R^{\otimes p})^{tC_p} \to R^{tC_p}$$

Using that in the multiplication map factors through the C_p-orbits.

Remark 20.11. We think of this as a spectral analogue of the ordinary Frobenius:

$$\begin{aligned} R &\to R/p \\ x &\mapsto x^p \end{aligned}$$

Exercise 20.12. *Check that his is a ringhomomorphism and that reducing mod p and R being commutative are necessary.*

Applied to $\mathrm{map}(\Sigma^\infty_+ X, E)$ we get the so called *Power Operations*.

21 The cyclotomic structure

For R a ring spectrum $THH(R) \in \mathrm{Sp}^{BS^1}$ and so we defined:

$$TC^-(R) = THH(R)^{hS^1},\ TP(R) = THH(R)^{tS^1}$$

this works in any stable co-complete symmetric monoidal ∞-category in place of Sp. No we introduce additional structure which is specific to spectra:
Construction:
Assume we have a fiber sequence:

$$X \to Y \to Z$$

of pointed connected spaces which we write as:

$$BH \to BG \to B(G/H)$$

Then there is a functor:

$$\begin{aligned} B(G/H) &\to \mathcal{S} \\ * &\mapsto BH \end{aligned}$$

which classifies BG, i.e. G/H acts on BH such that:

$$(BH)_{hG/H} \simeq BG$$

Exercise 21.1. *For a normal subgroup $H \subseteq G$ of a finite group, describe the G/H-action on BH obtained from the induced fiber sequence:*

$$BH \to BH \to B(G/H)$$

Consequence: For an ∞-category $\mathcal{C}$ we have:

$$\mathrm{Fun}(BG, \mathcal{C}) \simeq \mathrm{Fun}(BH, \mathcal{C})^{hG/H}$$

(With some additional work) this implies that the functors:

$$(-)^{hH},\ (-)^{tH} : \mathrm{Fun}(BH, \mathcal{C}) \to \mathcal{C}$$

induce functors:

$$(-)^{hH},\ (-)^{tH} : \mathrm{Fun}(BG, \mathcal{C}) = \mathrm{Fun}(BH, \mathcal{C}) \to \mathrm{Fun}(*, \mathcal{C})^{hG/H} = \mathrm{Fun}(B(G/H), \mathcal{C})$$

i.e. for a normal subgroup $H \subseteq G$ if X has a G-action, then $X^{hH},\ X^{tH}$ have a "residual" G/H-action such that:

$$(X^{hH})^{hG/H} \simeq X^{hG}$$

Example 21.2. Let $G = \mathbb{T} = U(1) = S^1$ and $H = C_p \subseteq \mathbb{T}$ the cyclic group of order p embedded as the roots of unity in $\mathbb{T}$. Then for $X \in \mathcal{C}^{B\mathbb{T}}$ we have that X^{hC_p} and X^{tC_p} have residual $\mathbb{T}/C_p$-actions. We identify:

$$\mathbb{T}/C_p \xrightarrow{\sim} \mathbb{T}$$
$$z \mapsto z^p$$

Definition 21.3. A *cyclotomic structure* on a spectrum with $\mathbb{T}$-action $X \in \mathrm{Sp}^{B\mathbb{T}}$ is given by maps:

$$\varphi_p : X \to X^{tC_p}$$

that are $\mathbb{T}$-equivariant, for every prime p. We refer to φ_p as the *cyclotomic Frobenius.*

- A *cyclotomic spectrum* is a spectrum with $\mathbb{T}$-action and cyclotomic structure.

Theorem 21.4. *For every ring spectrum R, $THH(R)$ admits a natural cyclotomic structure.*

Remark 21.5.
- The cyclotomic structure on $THH(R)$ is induced from the Tate-diagonal $\Delta_p : \mathrm{id} \to T_p$.
- It does not exists in general for $HH(R/\mathcal{C})$ where $\mathcal{C}$ is an arbitrary stable symmetric monoidal ∞-category. One can for example prove that ub $D(\mathbb{Z})$ there is no natural map:

$$X \to (X \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}})^{tC_p}$$

Therefore $HH(R/\mathbb{Z})$ does not admit any sort of cyclotomic structure.

First construction for an $\mathbb{E}_\infty - ring$: We need the following Tool called *induction*:
Let $\mathcal{C}$ be an ∞-category with all small colimits, then for any group $G \in \mathcal{S}$ we have a forgetful functor:

$$\mathcal{C}^{BG} \to \mathcal{C}$$

Proposition 21.6. *This functor has a left adjoint given by:*

$$c \in \mathcal{C} \mapsto \operatorname*{colim}_G(\mathrm{const}_c) = c \otimes G$$

with G-action G.

Proof. We have that:

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}^{Bg}}(c\otimes G,d) &\simeq \mathrm{Map}_{\mathcal{C}}(c\otimes G,d)^{hG}\\ &\simeq \mathrm{Map}_{\mathcal{S}}(G,\mathrm{Map}_{\mathcal{C}}(c,d))^{hG}\\ &\simeq \mathrm{Map}_{\mathcal{C}}(c,d)\end{aligned}$$

□

Example 21.7.

- $\mathcal{C} = \mathrm{Sp}$ and G finite, then $\bigoplus_{g\in G} c$ is the free object with G-action on c.
- For $G = \mathbb{T}$ we get $c \otimes \mathbb{T}c \oplus c[1]$

Let $\mathcal{C} = \mathrm{CAlg}(\mathrm{Sp})$ and G finite, $R \in \mathcal{C}$. then we have:

$$\operatorname*{colim}_G(R) = \bigotimes_G R = R^{\otimes G}$$

with the permutation $G - action$.

Proposition 21.8. *For $R \in \mathrm{CAlg}(\mathrm{Sp})$ we have that $R \to THH(R)$ exhibits $THH(R)$ as the free $\mathbb{E}_\infty$-ring with $\mathbb{T}$-action under R.*

Proof. By our general formula the free object is given by:

$$\operatorname*{colim}_{\mathbb{T}} R = R^{\otimes\mathbb{T}} = R \otimes_{R\otimes R} R = THH(R)$$

The fact that this $\mathbb{T}$-action agrees with the previously constructed via cyclic objects is omitted. (One observes that in the commutative cases one actually gets a cyclic object in $\mathbb{E}_\infty$-rings and the action is literally the on given on the simplicial object S^1). □

<u>Construction</u>:

- For any $R \in \mathrm{CAlg}(\mathrm{Sp})$ we have the map $R \to THH(R)$
- The target has a $\mathbb{T}$-action, in particular we get an induced C_p-action. Thus we get a unique extension to a C_p-equivariant map:

  $$\underbrace{R\otimes\ldots R}_{p\text{times}} \to THH(R)$$

  Exercise 21.9. *Check that this map agrees with the inclusion of the degree p part of the cyclic Bar complex.*

  This induces a map:
  $$(R\otimes\cdots\otimes R \to THH(R)^{tC_p})$$

- The Tate diagonal is a map of $\mathbb{E}_\infty$-rings:

  $$R \to (R\otimes\cdots\otimes R)^{tC_p}$$

Proposition 21.10. *There is a unique $\mathbb{T}$-equivariant map of $\mathbb{E}_\infty$-rings:*

$$THH(R) \xrightarrow{\varphi_p} THH(R)^{tC_p}$$

fitting into the diagram:

$$\begin{array}{ccc} R & \longrightarrow & THH(R) \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \exists!\varphi_p} \\ (R\otimes\cdots\otimes R)^{tC_p} & \longrightarrow & THH(R)^{tC_p}\end{array}$$

The case of $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$:
Recall that;

$$THH(R) = \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} (R^{\otimes n+1}) = \operatorname*{colim}_{\Delta^{\mathrm{op}}}(THH(R)_\bullet)$$

Where $THH(R)_\bullet : \Lambda \to \mathrm{Sp}$ is in fact a cyclic object. We want to derive the cyclotomic structure from the map of cyclic objects:

$$\begin{array}{ccccccc}
 & & \overset{C_3}{\circlearrowright} & & \overset{C_2}{\circlearrowright} & & \\
\cdots & \rightrightarrows & R\otimes R\otimes R & \rightrightarrows & R\otimes R & \longrightarrow & R \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_p} \\
\cdots & \rightrightarrows & (R^{\otimes 3p})^{tC_p} & \rightrightarrows & (R^{\otimes 2p})^{tC_p} & \longrightarrow & (R^{\otimes p})^{tC_p}
\end{array}$$

given by applying the Tate Diagonal to $R^{\otimes n}$. The main complication comes from the way of seeing the target as a cyclic object. This is denoted $(\mathrm{sd}_p THH(R))^{tC_p}$.

Exercise 21.11. *For an ordinary associative ring R, explicitly describe a cyclic object whose underlying simplicial object has degree K part* $(R^{\otimes p(k+1)})^{C_p}$

As a result we get a map of spectra with $\mathbb{T}$-action:

$$THH(R) = \operatorname*{colim}_{\Delta^{\mathrm{op}}}(THH(R)_\bullet) \to \operatorname*{colim}_{\Delta^{\mathrm{op}}}((\mathrm{sd}_p THH(R))^{tC_p})$$

$$\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \left((R^{\otimes p(n+1)})^{tC_p} \right) \to \left(\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} (R^{\otimes p(n+1)}) \right)^{tC_p} = THH(R)^{tC_p}$$

22 Definition of Topological Cyclic Homology

Recall that on Topological Hochschild Homology we had a *cyclotomic structure*, i.e. for every prime number p, we had a $\mathbb{T}$-equivariant map:

$$THH(R) \to THH(R)^{tC_p}$$

Definition 22.1. The ∞-category of cyclotomic spectra is the pullback:

$$\begin{array}{ccc}
\mathrm{CycSp} & \longrightarrow & \prod_p \mathrm{Sp}^{\Delta^1} \\
\downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)_p} \\
\mathrm{Sp}^{B\mathbb{T}} & \xrightarrow[(\mathrm{id},(-)^{tC_P})_p]{} & \prod_p \mathrm{Sp}^{B\mathbb{T}} \times \mathrm{Sp}^{B\mathbb{T}}
\end{array}$$

In particular a map of cyclotomic spectra (X, φ_p) to (Y, φ'_p) is given by:

1. A $\mathbb{T}$-equivariant map $X \xrightarrow{f} Y$
2. For every prime p a $\mathbb{T}$-equivariant homotopy filling the square:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \varphi_p}\downarrow & \swarrow & \downarrow{\scriptstyle \varphi'_p} \\
X^{tC_p} & \xrightarrow[f^{tC_p}]{} & Y^{tC_p}
\end{array}$$

The mapping space is given as the equalizer:

$$\mathrm{Map}_{\mathrm{CycSp}}(X,Y) \longrightarrow \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}(X,Y) \underset{(\varphi'_p)_*}{\overset{\varphi_p^*}{\rightrightarrows}} \prod_p \mathrm{Map}_{\mathrm{Sp}^{\mathbb{T}}}(X, Y^{tC_p})$$

Proposition 22.2. *1.* CycSp *is a stable ∞-category and the equalizer formula also holds for mapping spectra.*

2. CycSp *has all limits and colimits.*

3. CycSp *has a symmetric monoidal structure given by:*

$$(X,\varphi_p) \otimes (Y,\varphi'_p) := (X \otimes_{\mathbb{S}} Y), X \otimes_{\mathbb{S}} Y \to X^{tC_p} \otimes_{\mathbb{S}} Y^{tC_p} \to (X \otimes Y)^{tC_p}$$

where we use that the Tate construction is lax symmetric monoidal.

4. A commutative algebra in CycSp *is given by:*

(a) $X \in \mathrm{CAlg}(\mathrm{Sp})^{B\mathbb{T}}$

(b) $\mathbb{T}$*-equivariant maps of commutative algebra objects:*

$$\varphi_p : X \to X^{tC_p}$$

In particular for an $\mathbb{E}_\infty$-ring is a commutative algebra in CycSp

Exercise 22.3. *Show the first part*

Recall: We had defined:

$$TC^-(R) := THH(R)^{hS^1} = \mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(\mathbb{S}^{\mathrm{triv}}, THH(R))$$

And we will five a similar description for TC!

Example 22.4. $\mathbb{S}^{\mathrm{triv}}$ is canonically a cyclotomic spectrum ($\mathbb{E}_\infty$):

- The underlying spectrum with $\mathbb{T}$-action is $\mathbb{S}^{\mathrm{triv}}$
- For every p the map:

$$\mathbb{S} \to \mathbb{S}^{tC_p}$$

  is the unit, i.e. the map:

$$\mathbb{S} \to \mathbb{S}^{hC_p} \xrightarrow{\mathrm{can}} \mathbb{S}^{tC_p}$$

  where the first map is the pullback of $BC_p \to *$.

This is in fact the tensor unit in CycSp.

Exercise 22.5. *Lift the factorisation $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$ to a diagram of $\mathbb{T}$-equivariant maps.*

Definition 22.6. – For a ring spectrum R we define:

$$TC(R) := \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, THH(R))$$

– If R is commutative, then this is an $\mathbb{E}_\infty$-ring sppectrum
We will write $TC_*(R) = \pi_* TC(R)$ as usual.

More general for any cyclotomic spectrum X we write:

$$TC(X) := \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X)$$

so that in particular $TC(R) = TC(THH(R))$.

Let's explicitly evaluate this formula:

$$
\begin{aligned}
TC(X) = \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X) &\simeq \mathrm{Eq}\left(\mathrm{map}_{\mathrm{Sp}^{\mathbb{T}}}(\mathbb{S}^{\mathrm{triv}}, X) \rightrightarrows \prod_p \mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(\mathbb{S}^{\mathrm{triv}}, X^{tC_p})\right) \\
&\simeq \mathrm{Eq}\left(X^{h\mathbb{T}} \rightrightarrows \prod_p \left(X^{tC_p}\right)^{h\mathbb{T}}\right) \\
&\simeq \mathrm{fib}\left(X^{h\mathbb{T}} \xrightarrow{\varphi_p^{h\mathbb{T}} - \mathrm{can}} \prod_p \left(X^{tC_p}\right)^{h\mathbb{T}}\right)
\end{aligned}
$$

with $\varphi_p^{h\mathbb{T}} : X^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$ and can given by the induced map $X^{h\mathbb{T}} = (X^{hC_p})^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$

Theorem 22.7. *Assume that X is bounded below with $\mathbb{T}$-action, then the canonical map:*

$$X^{t\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$$

exhibits the right hand side as the p-completion of the left hand side. Moreover if X is p-complete then so is $X^{t\mathbb{T}}$.

Corollary 22.8. *If $X \in \mathrm{CycSp}$ with underlying spectrum bounded below, then:*

$$\varphi_p^{h\mathbb{T}} : X^{h\mathbb{T}} \to \left(X^{tC_p}\right)^{h\mathbb{T}} = \left(X^{t\mathbb{T}}\right)_p^{\wedge}$$

$$\varphi : X^{h\mathbb{T}} \to \left(X^{t\mathbb{T}}\right)^{\wedge} := \prod_p \left(X^{t\mathbb{T}}\right)_p^{\wedge}$$

So we get that:

$$TC(X) \simeq \mathrm{fib}\left(X^{h\mathbb{T}} \xrightarrow{\varphi - \mathrm{can}} \left(X^{t\mathbb{T}}\right)^{\wedge}\right)$$

And in particular for a connective ring spectrum R:

$$TC(R) \simeq \mathrm{fib}\left(TC^{-}(R) \xrightarrow{\varphi - \mathrm{can}} TP(R)^{\wedge}\right)$$

For X a qcsqs scheme we define:

$$THH(X) := \lim_{U \subseteq X \text{affine open}} THH(\mathcal{O}_X)$$

$$TC^{-}(X) := THH(X)^{h\mathbb{T}} = \lim_{U \subseteq X \text{ affine open}} TC^{-}(\mathcal{O}_X)$$

$$TP(X) := THH(X)^{t\mathbb{T}} = \lim_{U \subseteq X \text{ affine open}} TP(\mathcal{O}_X)$$

$$TC(X) := TC(THH(X)) \simeq \lim_{U \subseteq X} TC(\mathcal{O}_U) \simeq \mathrm{Eq}\left(TC^{-}(X) \rightrightarrows TP(X)^{\wedge}\right)$$

Remark 22.9. one can show, that for $X \in \mathrm{CycSp}$ n-connective that:

$$TC(X)_p^{\wedge} = TC(X, \mathbb{Z}_p)$$

is $n-1$-connective. In particular $TC(R, \mathbb{Z}_p)$ for R a ring or connective ring spectrum, is -1-connective.