ON THE DEFORMATION THEORY OF $\mathbb{E}_\infty$-COALGEBRAS

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Abstract. We introduce a notion of formally étale $\mathbb{E}_\infty$-coalgebras and show that these admit essentially unique, functorial lifts against square zero extensions of $\mathbb{E}_\infty$-rings. We use this to construct a spherical Witt Vector style functor which exhibits the ∞-category of formally étale connective $\mathbb{E}_\infty$-coalgebras over $\mathbb{F}_p$ as a full subcategory of the ∞-category of p-complete connective $\mathbb{E}_\infty$-coalgebras over $\mathbb{S}_p^\wedge$. Finally, we prove that for a finite space X the $\mathbb{F}_p$-homology $\mathbb{F}_p[X] = C_*(X;\mathbb{F}_p)$ is a formally étale $\mathbb{F}_p$-coalgebra and hence $\mathbb{S}[X]_p^\wedge = (\Sigma_+^\infty X)_p^\wedge$ can be recovered its essentially unique lift to the p-completed sphere.

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1. Introduction

One of the fundamental goals of algebraic topology is to construct algebraic models for homotopy types, that is to give complete homotopy invariants for topological spaces. While this is unrealistic for all spaces simultaneously, a divide-and-conquer approach borrowed from arithmetic has been immensely successful. Namely, we can approximate a space X by its rationalization $X_\mathbb{Q}$ and for each prime p its p-completion $X_p^\wedge$. The rational story has been well understood for many years due to work of Sullivan [Sul77] and Quillen [Qui69]. We are interested in the p-adic setting, where a theorem of Mandell shows that we can model a full subcategory of the ∞-category of p-complete spaces $\mathcal{S}_p^\wedge$ using $\mathbb{E}_\infty$-algebras.

Theorem 1.1 (Mandell [Man01])**.** *The assignment $X \mapsto C^*(X;\overline{\mathbb{F}}_p) = X^{\overline{\mathbb{F}}_p}$ determines a fully faithful contravariant functor from the full subcategory of $\mathcal{S}_p^\wedge$ spanned by the simply*

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connected p-complete spaces of finite type to the ∞-category $\mathrm{CAlg}_{\overline{\mathbb{F}}_p}$ *of $\mathbb{E}_\infty$-algebras over $\overline{\mathbb{F}}_p$.*

In particular, this tells us that the space X can be recovered as the space of $\mathbb{E}_\infty$-ring maps $\overline{\mathbb{F}}_p^X \to \overline{\mathbb{F}}_p$, hence the $\overline{\mathbb{F}}_p$-cohomology remembers everything there is to know about the space X. However, this theorem raises some questions. Firstly, the $\mathbb{E}_\infty$-algebra structure on $X^{\overline{\mathbb{F}}_p}$ is actually induced via dualizing by the $\mathbb{E}_\infty$-coalgebra structure on the $\overline{\mathbb{F}}_p$-homology $C_*(X;\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p[X]$. Since the duality functor

$$\mathrm{cCAlg}_{\overline{\mathbb{F}}_p} \xrightarrow{(-)^\vee} \mathrm{CAlg}_{\overline{\mathbb{F}}_p}^{\mathrm{op}}$$

is not fully faithful, it is natural to ask whether we can exhibit Mandell's Theorem as a shadow of an even better coalgebraic model for p-complete spaces. We can also wonder why the ring $\overline{\mathbb{F}}_p$ in Theorem 1.1 appears instead of $\mathbb{F}_p$. Indeed, the $\mathbb{F}_p$-cohomology functor $X \mapsto \mathbb{F}_p^X$ is *not* fully faithful because we are forgetting the Frobenius action on the coefficients. More precisely, Mandell shows that

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{F}_p}}(\mathbb{F}_p^X, \mathbb{F}_p) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\overline{\mathbb{F}}_p}}(\overline{\mathbb{F}}_p^X, \overline{\mathbb{F}}_p)^{h\mathbb{Z}} \simeq X^{h\mathbb{Z}} \simeq \mathcal{L}X$$

holds, where $\mathbb{Z}$ acts via the Frobenius on $\overline{\mathbb{F}}_p$. This tells us that any algebraic model of p-complete spaces needs to take a Frobenius into account. For the $\mathbb{E}_\infty$-algebra model we can summarize this story informally as follows.

Slogan. *The datum of a simply connected p-complete space is equivalent to that of an $\mathbb{E}_\infty$-ring together with a trivialization of the Frobenius via the cohomology functor.*

To formalize this one would like to use a notion of Frobenius that is intrinsic to the category of $\mathbb{E}_\infty$-algebras and not only defined via the coefficients. Such a notion is provided by the Tate Frobenius map

$$\varphi_p : R \to R^{hC_p} \to R^{tC_p}$$

introduced by Nikolaus and Scholze in [NS17] for every $\mathbb{E}_\infty$-algebra R, which is a generalization of the p-th power map $R \to R/p$ which takes values in the *Tate Homology* of R. Unlike the ordinary p-th power map, it is not an endomorphism over $\mathbb{F}_p$ since $\mathbb{F}_p^{tC_p}$ is not equivalent to $\mathbb{F}_p$. However, it is much better behaved over the p-completed sphere since $(\mathbb{S}_p^\wedge)^{tC_p} \simeq \mathbb{S}_p^\wedge$ does hold by the Segal Conjecture, first proved in this form for $p = 2$ by Lin in [Lin80] and for odd p by Gunawardena in [Gun80]. This suggests instead considering $\mathbb{S}_p^{\wedge X}$ as a model for a p-complete homotopy type X. These ideas were recently investigated by Yuan in [Yua19] to give a model for p-complete finite spaces in terms of so called *p-Frobenius fixed* $\mathbb{E}_\infty$*-rings*, see [Yua19, Theorem 7.1.]. This is a natural extension of the $\mathbb{F}_p$-model as a consequence of the following statement.

Proposition 1.2 (Mandell, Lurie)**.** *For any space X the $\mathbb{F}_p$-cohomology $\mathbb{F}_p^X$ is a formally étale $\mathbb{F}_p$-algebra.*

Thus, one can use deformation theoretic arguments to show that for every finite space X the base change map

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(\mathbb{S}_p^{\wedge X}, \mathbb{S}_p^\wedge) \to \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{F}_p}}(\mathbb{F}_p^X, \mathbb{F}_p)$$

is a homotopy equivalence, compare the argument in [Yua19, Corollary 7.6.1.]. This crucially uses that $\mathbb{S}_p^{\wedge X} \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p \simeq \mathbb{F}_p^X$ for finite X. The finiteness assumption is also necessary for the model to ensure that the Frobenius defines an automorphism on $\mathbb{S}_p^{\wedge X}$. Thus, if we want to improve on this result and fully realize our slogan, we need to find better behaved Frobenius map. As it turns out, this is available to us if we are willing to work with p-complete $\mathbb{E}_\infty$-*coalgebras* instead as is the content of the following unpublished result due to Nikolaus.

Theorem 1.3 (Nikolaus)**.** *Let $\mathcal{C} = (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})_p^\wedge$ denote the ∞-category $\mathbb{E}_\infty$-coalgebras in the category of p-complete spectra equipped with the symmetric monoidal structure given by the p-completed tensor product. Then there exists a natural transformation $\psi_p : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ which on an object $A \in \mathcal{C}$ is given by the composition*

$$\psi_p : A \xrightarrow{\Delta_A^{\otimes p}} (A^{\otimes p})^{hC_p} \xrightarrow{\mathrm{can}} (A^{\otimes p})^{tC_p} \xrightarrow{\sim} A,$$

where the right hand map is the inverse of the Tate Diagonal, see [NS17, Theorem III.1.7].

This means that for p-complete $\mathbb{S}_p^\wedge$-coalgebra A we always have access to a Frobenius *endomorphism* $\psi_p : A \to A$. Moreover, the homology of a space is better behaved with respect to base change, as we have $\mathbb{S}_p[X]_p^\wedge \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p \simeq \mathbb{F}_p[X]$ for *any* space X. This suggests instead considering $\mathbb{S}[X]_p^\wedge = (\Sigma^\infty_+ X)_p^\wedge$ together with the action of ψ_p as a more natural model for p-complete spaces X. To obtain a coalgebraic version of Theorem 1.1 from this we would have to show that the base change map

$$\mathrm{Map}_{(\mathrm{cCAlg}_{\mathbb{S}_p^\wedge})_p^\wedge}(\mathbb{S}_p^\wedge, \mathbb{S}[X]_p^\wedge) \to \mathrm{Map}_{\mathrm{cCAlg}_{\mathbb{F}_p}}(\mathbb{F}_p, \mathbb{F}_p[X])$$

is an equivalence. This is the main problem this paper concerns itself. Put differently, we can divide the problem of understanding the $\mathbb{F}_p$-homology as a model for p-complete spaces into two parts by factoring the functor $X \mapsto \mathbb{F}_p[X]$ as follows

$$\mathcal{S} \xrightarrow{\mathbb{S}[-]_p^\wedge} (\mathrm{cCAlg}_{\mathbb{S}_p^\wedge})_p^\wedge \xrightarrow{-\otimes_{\mathbb{S}_p^\wedge}\mathbb{F}_p} \mathrm{cCAlg}_{\mathbb{F}_p}.$$

We want to understand the right hand functor. More concretely we ask the following:

Question 1.4. *How can we describe the full subcategory of $(\mathrm{cCAlg}_{\mathbb{S}_p^\wedge})_p^\wedge$ spanned by those $\mathbb{E}_\infty$-coalgebras for which the base change to $\mathbb{F}_p$ is fully faithful? Moreover, does it contain $\mathbb{S}[X]_p^\wedge$ for an arbitrary space X?*

Since the base change to $\mathbb{F}_p$ can further be factored as the composition

$$(\mathrm{cCAlg}_{\mathbb{S}_p^\wedge})_p^\wedge \xrightarrow{-\otimes_{\mathbb{S}_p^\wedge}\mathbb{Z}_p} (\mathrm{cCAlg}_{\mathbb{Z}_p})_p^\wedge \xrightarrow{-\otimes_{\mathbb{Z}_p}\mathbb{F}_p} \mathrm{cCAlg}_{\mathbb{F}_p}$$

this question can be phrased in terms of *deformation theory*, which is the approach taken by this paper. More precisely, since $\mathbb{Z}_p$ is an iterated square zero extension of $\mathbb{F}_p$ and in turn $\mathbb{S}_p^\wedge$ is an iterated square zero extension of $\mathbb{Z}_p$, it is natural to first ask how to lift $\mathbb{E}_\infty$-coalgebras along general square zero extensions. This is also called a *deformation problem.*

Question 1.5. *Let R be an $\mathbb{E}_\infty$-ring, $R^\eta \to R$ be a square zero extension and $A \in \mathrm{cCAlg}_R$. How can we describe the space of R^η-coalgebras A' equipped with an equivalence $A' \otimes_{R^\eta} R \simeq A$? For which coalgebras is it contractible?*

1.1. **Summary of Results.** To answer Questions 1.4 and 1.5 we introduce a novel and somewhat surprising notion of *formally étale* coalgebras. Let R be an $\mathbb{E}_\infty$-ring and $M \in \mathrm{Mod}_R$. Denote by $f_M : R \to R \oplus M$ the section of the split square zero extension and write $f_M^* : \mathrm{cCAlg}_R \to \mathrm{cCAlg}_{R\oplus M}$ for the base change functor. This functor admits a right adjoint by the adjoint functor theorem which we denote $f_{M,!}$.

Definition 1.6. A coalgebra $A \in \mathrm{cCAlg}_R$ is called *formally étale* if the counit of the adjunction $\eta_A : A \to f_{M,!}f_M^*A$ is an equivalence for every $M \in \mathrm{Mod}_R$. We denote by $\mathrm{cCAlg}_R^{\mathrm{f\acute{e}t}}$ the full subcategory spanned by the formally étale coalgebras.

The relation to deformation theory is as follows: The counit admits a natural splitting $\pi_A : f_{M,!}f_M^*A \to A$, hence η_A is an equivalence if and only if π_A is. We show that all relevant deformation problems are equivalent to lifting problems of the form

$$\begin{array}{ccc} & & f_{M,!}f_M^*A \\ & \nearrow & \downarrow{\scriptstyle \pi_A} \\ B & \longrightarrow & A \end{array}$$

in the category of R-coalgebras. If we ask that these can be solved uniquely for every $B \in \mathrm{cCAlg}_R$, the Yoneda Lemma implies that π_A and thus also η_A must be an equivalence. This may seem strange and unreasonable at first glance. However, notice that since any $\mathbb{E}_\infty$-ring is the terminal coalgebra over itself we have

$$f_{M,!}f_M^*(R) \simeq f_{M,!}(R \oplus M) \simeq R$$

as $f_{M,!}$ is a right adjoint and hence preserves terminal objects. More generally we show that the following holds.

Proposition 1.7. *Let $A \in \mathrm{cCAlg}_R$ be dualizable such that its dual $A^\vee$ is a formally étale R-algebra. Then A is formally étale in the sense of Definition 1.6.*

This shows that Definition 1.6 is actually *reasonable* i.e. it is satisfied by a nontrivial class of coalgebras. We also prove that it is *powerful*, namely that formally étale coalgebras can be lifted uniquely and functorially along square zero extensions.

Proposition 1.8. *Let R be a connective $\mathbb{E}_\infty$-ring, $R^\eta \to R$ a square zero extension and denote by $\mathcal{C} \subseteq \mathrm{cCAlg}^{\mathrm{cn}}_{R^\eta}$ the full subcategory spanned by those coalgebras A such that $A \otimes_{R^\eta} R$ is formally étale. Then the functor*

$$\mathcal{C} \to \mathrm{cCAlg}^{\mathrm{cn,fét}}_{\mathbb{F}_p} \qquad A \mapsto A \otimes_{R^\eta} R$$

is fully faithful and essentially surjective. In particular, the quasi-inverse defines a functor

$$W_\eta : \mathrm{cCAlg}^{\mathrm{cn,fét}}_{R} \to \mathrm{cCAlg}^{\mathrm{cn}}_{R^\eta}$$

which is fully faithful and satisfies $W_\eta(A) \otimes_{R^\eta} R \simeq A$. Moreover, up to contractible choice $W_\eta(A)$ is the unique connective R^η coalgebra with this property.

Our main theorem is an upgrade of this statement to a Spherical Witt-Vector Style functor for $\mathbb{E}_\infty$-coalgebras over $\mathbb{F}_p$.

Theorem 1.9. *Denote by $\mathcal{C} \subseteq (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$ the full subcategory spanned by those coalgebras A such that $A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$ is formally étale. Then the functor*

$$\mathcal{C} \to \mathrm{cCAlg}^{\mathrm{cn,fét}}_{\mathbb{F}_p} \qquad A \mapsto A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$$

is fully faithful and essentially surjective. In particular, the quasi-inverse defines a functor

$$W_{\mathbb{S}_p^\wedge} : \mathrm{cCAlg}^{\mathrm{cn,fét}}_{\mathbb{F}_p} \to (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$$

which is fully faithful and satisfies $W_{\mathbb{S}_p^\wedge}(A) \otimes_{\mathbb{S}_p} \mathbb{F}_p \simeq A$. Moreover, up to contractible choice $W_{\mathbb{S}_p^\wedge}(A)$ is the unique connective p-complete $\mathbb{S}_p^\wedge$-coalgebra with this property.

For a finite space X the coalgebra $\mathbb{F}_p[X]$ is dualizable with dual given by $\mathbb{F}_p^X$. Thus, since $\mathbb{F}_p^X$ is formally étale we can combine Proposition 1.7 and Theorem 1.9 to get a partial answer to our questions.

Proposition 1.10. *For a finite space X the coalgebra $\mathbb{S}[X]^\wedge_p$ is the essentially unique lift of $\mathbb{F}_p[X]$ to a p-complete, connective $\mathbb{S}_p^\wedge$-coalgebra. Moreover, for any finite spaces X, Y the base change map*

$$\mathrm{Map}_{(\mathrm{cCAlg}_{\mathbb{S}_p^\wedge})^\wedge_p}(\mathbb{S}[X]^\wedge_p, (\mathbb{S}[Y]^\wedge_p) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{cCAlg}_{\mathbb{F}_p}}(\mathbb{F}_p[X], \mathbb{F}_p[Y])$$

is a homotopy equivalence.

Since this notion of formally étale is defined using the mysterious right adjoint $f_{M,!}$, what is most crucially still missing is a sufficient condition for a non-dualizable coalgebra to be formally étale that can be checked in practice. We conjecture that this is provided by the coalgebra Frobenius discussed above.

Conjecture 1.11. *Let $A \in (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$, then for any $M \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p}$, the coalgebra Frobenius $\psi_p : A \to A$ induces the zero map on the R-module $C_{A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p}(M) = \mathrm{cofib}(A \xrightarrow{\eta_A} \Omega^\infty_A(M))$.*

This would immediately imply that, if A is a p-complete $\mathbb{S}_p^\wedge$-coalgebra such that the Frobenius $\psi : A \to A$ is a homotopy equivalence, then $A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$ is a formally étale $\mathbb{F}_p$-coalgebra. Moreover, since by naturality the coalgebra Frobenius is given by the identity on $\mathbb{S}[X]_p^\wedge$, it would allow us to apply our theorem to the chains of spaces that are not-necessarily finite and supply us with examples of formally étale coalgebras which are not dualizable.

1.2. **Technical methods.** We attack these problems using the machinery of deformation theory developed by Lurie in [Lur12] and [Lur17]. In [Lur12] a class of functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is introduced which are well behaved with respect to deformation problems.

Definition 1.12 (Lurie)**.** A functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is called *cohesive* if for any pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

which induces surjections $\pi_0 R \to \pi_0 S$ and $\pi_0 S' \to \pi_0 S$, the diagram

$$\begin{array}{ccc} X(R') & \longrightarrow & X(S') \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(S) \end{array}$$

is a pullback of spaces.

Given a cohesive functor X for each R-valued point $A \in X(R)$ there exists a spectrum $T_{X_A}^M$ which controls deformations of A along square zero extensions of R with fiber M.

Theorem 1.13 (Lurie)**.** *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a cohesive functor and $R^\eta \to R$ a square zero extension with fiber $M \in \mathrm{Mod}_R^{\mathrm{cn}}$. Then for each $A \in X(R)$ there exists a spectrum $T_{X_A}^M$ called the Tangent Complex of X at A such that the space of deformations $\mathrm{X}_A^{R^\eta} = \mathrm{fib}_A(X(R^\eta) \to X(R))$ is either empty or a torsor under the grouplike $\mathbb{E}_\infty$-monoid $\Omega^\infty T_{X_A}^M$. Moreover, we have an obstruction class in $\pi_{-1} T_{X_A}^M$ which vanishes if and only if $\mathrm{X}_A^{R^\eta}$ is non-empty.*

In this sense, the tangent complex, if it exists, is the precise answer to Question 1.5. We use an étale descent theorem for modules also due to Lurie [Lur21, Theorem 16.2.0.2.] to show that this machinery can be applied to coalgebras.

Proposition 1.14. *For any $n \in \mathbb{N}$ the functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which takes a connective $\mathbb{E}_\infty$-ring R to the space $(\mathrm{cCAlg}_R^{\mathrm{cn}})^{\Delta^n} = \mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^n, \mathrm{cCAlg}_R^{\mathrm{cn}})$ is cohesive.*

This allows us to make sense of the definition of formally étale coalgebras and prove Proposition 1.8. To get from this to Theorem 1.9, we prove two different completeness results for coalgebras. Concretely, we can write the ring $\mathbb{Z}_p$ as the limit

$$\mathbb{Z}_p = \lim\left(\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p = \mathbb{F}_p\right)$$

where each map is a square zero extension with fiber $\mathbb{Z}/p$. This poses a problem, as the natural map

$$\mathrm{cCAlg}_{\mathbb{Z}_p} \to \varprojlim \mathrm{cCAlg}_{\mathbb{Z}/p^n}$$

is *not* an equivalence. This is where *p-complete* coalgebras come in to play.

Definition 1.15. Let R be an $\mathbb{E}_\infty$-ring. We define the ∞-category of p-complete R-coalgebras as

$$(\mathrm{cCAlg}_R)^\wedge_p := \mathrm{cCAlg}((\mathrm{Mod}_R)^\wedge_p).$$

Here, $(\mathrm{Mod}_R)^\wedge_p$ denotes the ∞-category of p-complete R-modules equipped with the symmetric monoidal structure given by the p-completed tensor product. We prove that p-complete coalgebras are suitable for deformation theoretic questions and in fact the correct notion if we want to pass from $\mathbb{F}_p$ to the p-adics inductively.

Proposition 1.16. *For every $n \in \mathbb{N}$ the functor*

$$\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \qquad R \mapsto [(\mathrm{cCAlg}_R)^\wedge_p]^{\Delta^n}$$

is cohesive. Moreover, the assignment $A \mapsto A \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n$ induces an equivalence of ∞-categories

$$(\mathrm{cCAlg}_{\mathbb{Z}_p})^\wedge_p \xrightarrow{\sim} \varprojlim \mathrm{cCAlg}_{\mathbb{Z}/p^n}.$$

Similarly, $\mathbb{S}^\wedge_p$ is given by the limit of the Postnikov-Tower

$$\mathbb{S}^\wedge_p = \lim\left(\cdots \to \tau_{\le 2}\mathbb{S}^\wedge_p \to \tau_{\le 1}\mathbb{S}^\wedge_p \to \tau_{\le 0}\mathbb{S}^\wedge_p = \mathbb{Z}_p\right),$$

where each map $\tau_{\le n+1}\mathbb{S}^\wedge_p \to \tau_{\le n}\mathbb{S}^\wedge_p$ is a square zero extension with fiber $\pi_{n+1}\mathbb{S}^\wedge_p[n+1]$. Thus, to be able to lift inductively from $\mathbb{Z}_p$ to $\mathbb{S}^\wedge_p$ we prove the following.

Proposition 1.17. *For every connective $\mathbb{E}_\infty$-ring R, the truncation functors $\tau_{\le n} : \mathrm{Mod}_R \to \mathrm{Mod}_{\tau_{\le n}R}$ induce equivalences of categories*

$$\mathrm{cCAlg}^{\mathrm{cn}}_R \xrightarrow{\sim} \varprojlim \mathrm{cCAlg}^{\mathrm{cn}}_{\tau_{\le n}R}$$
$$(\mathrm{cCAlg}^{\mathrm{cn}}_R)^\wedge_p \to \varprojlim_n (\mathrm{cCAlg}^{\mathrm{cn}}_{\tau_{\le n}R})^\wedge_p.$$

With these requirements in place, Theorem 1.9 follows from abstract properties of the tangent complex and its interaction with the formally étale property.

1.3. **Outline.** We proceed along the following structure: In Section 1 we define our basic objects of study and collect some facts about coalgebras and duality. In Section 2 we introduce and review the setup of deformation theory developed by Lurie in [Lur12] and [Lur17]. We recall the notions of square zero extensions of $\mathbb{E}_\infty$-rings and discuss cohesive and nilcomplete functors. We then define the (co)tangent complex of a cohesive functor and prove some facts about its behavior. In Section 3 we investigate how to lift coalgebras and maps of coalgebras against square zero extensions. We apply the tangent complex formalism to deformations of maps of coalgebras and show that these deformation problems can be reformulated as a lifting problem against certain maps of coalgebras and use this to introduce our notion of formally étale coalgebras. In Section 4 we discuss how to lift coalgebras and morphisms of coalgebras from $\mathbb{F}_p$ to $\mathbb{S}_p^\wedge$. We construct a spherical Witt vector style functor for formally étale F_p-coalgebras and apply our results to $\mathbb{F}_p[X]$ for a finite space X. In Section 5 we give a brief overview of some unanswered questions and sketch a possible way to proceed with the program envisioned by Nikolaus.

1.4. **Conventions.** Throughout the text, we use the following conventions:

- We use the words category, ∞-category and $(\infty, 1)$-category interchangeably to mean $(\infty, 1)$-category. Moreover, the text is *model agnostic*, that is we make no reference to any specific model for the theory of $(\infty, 1)$-categories. If pressed, we fall back on the Weak Kan Complex model as described in [Lur08].
- If A, B are objects in some ∞-category $\mathcal{C}$, we use the words map and morphism $A \to B$ interchangeably to mean a point in the mapping space $\mathrm{Map}_{\mathcal{C}}(A, B)$.
- We denote the ∞-category of spaces by $\mathcal{S}$ and the ∞-category of spectra by Sp. Moreover, we write $\mathrm{Sp}_{\geq n} \subseteq \mathrm{Sp}$ for the full subcategory spanned by the n-connective spectra and denote the right adjoint to the inclusion by $\tau_{\geq n} : \mathrm{Sp} \to \mathrm{Sp}_{\geq n}$.
- We choose a Grothendieck Universe $\mathcal{U}$, denote by Cat_∞ the large ∞-category of small ∞-categories and disregard all size issues from here on out.
- By (co)algebra we always mean $\mathbb{E}_\infty$-(co)algebra.
- All tensor products are derived unless stated otherwise.

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2. Coalgebras

In this section, we collect some general definitions and facts which we will use throughout the text. We define $\mathbb{E}_\infty$-coalgebras in an arbitrary symmetric monoidal category $\mathcal{C}$ and discuss how well behaved this notion is with respect to colimits and presentability. We then specialize to the case $\mathcal{C} = \mathrm{Mod}_R$ and introduce the right adjoint to base change in the coalgebra setting. The latter will play a crucial and annoying role in the considerations of deformation theory that follow. We then move on to coalgebras in presentably symmetric monoidal ∞-categories, introducing the coalgebra structure on the R-chains of a space X

and the algebra structure on $A^\vee = \mathrm{map}_{\mathcal{C}}(A, 1_{\mathcal{C}})$ for $A \in \mathrm{cCAlg}(\mathcal{C})$ and $\mathcal{C}$ a presentably monoidal ∞-category. We begin by recalling basic facts about $\mathbb{E}_\infty$-algebras in an arbitrary symmetric monoidal ∞-category, before moving on to the coalgebra picture.

2.1. **Generalities.**

Definition 2.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, then we denote by $\mathrm{CAlg}(\mathcal{C})$ the category of $\mathbb{E}_\infty$-algebras in $\mathcal{C}$. For $\mathcal{C} = \mathrm{Sp}$ we write $\mathrm{CAlg}(\mathrm{Sp}) = \mathrm{CAlg}$ and refer to objects of CAlg simply as $\mathbb{E}_\infty$-rings.

Proposition 2.2 (Lurie)**.** *Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal ∞-category, then the following statements hold:*

1. *The forgetful functor $U : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is conservative and commutes with limits.*
2. *The coproduct of two algebras $R, S \in \mathrm{CAlg}(\mathcal{C})$ is given by the tensor product $R \otimes S$.*
3. *If $\mathcal{C}$ is presentable and $- \otimes -$ commutes with colimits in both variables separately, then $\mathrm{CAlg}(\mathcal{C})$ is presentable as well.*

Proof. The first claim is a combination of [Lur17, Lemma 3.2.2.6] and [Lur17, Corollary 3.2.2.5]. The second is shown in [Lur17, Corollary 3.2.4.7] and the third is [Lur17, Corollary 3.2.3.5.]. □

Definition 2.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and denote by $p : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ the associated ∞-operad. Composing the straightening of p with the opposite category functor $(-)^{\mathrm{op}} : \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ and taking the unstraightening yields a new ∞-operad $\mathrm{Un}(p^{\mathrm{op}}) : \mathcal{D}^{\otimes} \to \mathrm{Fin}_*$ with fiber over $\langle 1 \rangle \in \mathrm{Fin}_*$ given by $\mathcal{C}^{\mathrm{op}}$. This equips $\mathcal{C}^{\mathrm{op}}$ with a natural symmetric monoidal structure and we define the category of $\mathbb{E}_\infty$-coalgebras in $\mathcal{C}$ as

$$\mathrm{cCAlg}(\mathcal{C}) := \mathrm{CAlg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}.$$

Remark 2.4. In particular, any coalgebra $A \in \mathrm{cCAlg}(\mathcal{C})$ comes equipped with a datum of "coherently commutative" multiplication and counit maps

$$\Delta_A : A \to A \otimes A$$
$$\eta : A \to 1_{\mathcal{C}},$$

where $1_{\mathcal{C}}$ denotes the tensor unit of $\mathcal{C}$. Note that, in general, there is no way to describe $\mathrm{cCAlg}_{\mathcal{C}}$ as a category of algebras in some suitable category $\mathcal{D}$. Thus, coalgebras behave very differently from algebras, although they are still "well behaved" in many ways. For once, Proposition 2.2 immediately yields that for a symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ we have:

1. The forgetful functor $U : \mathrm{cCAlg}(\mathcal{C}) \to \mathcal{C}$ is conservative and commutes with colimits.
2. The product of two coalgebras $R, S \in \mathrm{cCAlg}(\mathcal{C})$ is given by $R \otimes S$.

However, we cannot deduce presentability this way since the opposite of a presentable category is almost never presentable.

Proposition 2.5 (Lurie)**.** *Let* $(\mathcal{C}, \otimes)$ *be a symmetric monoidal* ∞*-category such that* $-\otimes-$ *commutes with colimits in each variable separately and* $\mathcal{C}$ *is presentable. Then* $\mathrm{cCAlg}(\mathcal{C})$ *is presentable.*

Proof. This is [Lur18, Corollary 3.1.4.]. □

This can be seen as an analogue of the classical theorem by Sweedler that every coalgebra in the 1-category of vector spaces over a field is a filtered colimit of its finite dimensional sub-coalgebras, see [Swe69]. However, unlike in the classical situation, if $\mathcal{C}$ is κ-presentable, we only deduce that $\mathrm{cCAlg}(\mathcal{C})$ is τ-presentable for some unknown $\tau \geq \kappa$. This is one of the main defects the category of coalgebras has over that of algebras. Otherwise they are similarly well behaved.

Lemma 2.6. *Let* $\mathrm{Cat}_\infty^{\otimes}$ *denote the* ∞*-category of symmetric monoidal* ∞*-categories and strong monoidal functors. Then the functors*

$$\mathrm{Cat}_\infty^{\otimes} \to \mathrm{Cat}_\infty \quad \mathcal{C} \mapsto \mathrm{CAlg}(\mathcal{C}),$$

$$\mathrm{Cat}_\infty^{\otimes} \to \mathrm{Cat}_\infty \quad \mathcal{C} \mapsto \mathrm{cCAlg}(\mathcal{C})$$

commute with limits.

Proof. By Proposition 2.2 the functor $\mathrm{CAlg}(-)$ factors through the category $\mathrm{Cat}_\infty^{\amalg}$ of categories which admit finite coproducts and functors which preserve finite coproducts. As such it admits a left adjoint which equips $\mathcal{D} \in \mathrm{Cat}_\infty^{\amalg}$ with the cocartesian monoidal structure. Moreover, the inclusion $\mathrm{Cat}_\infty^{\amalg} \hookrightarrow \mathrm{Cat}_\infty$ admits a left adjoint which takes an ∞-category $\mathcal{C}$ to the free finite-coproduct completion, namely the full subcategory of $\mathrm{Psh}(\mathcal{C})$ spanned by finite coproducts of representables. Thus, both functors commute with limits, and so the composition does as well.
For the second functor, we simply observe that it is given by the composition

$$\mathrm{Cat}_\infty^{\otimes} \xrightarrow{(-)^{\mathrm{op}}} \mathrm{Cat}_\infty^{\otimes} \xrightarrow{\mathrm{CAlg}(-)} \mathrm{Cat}_\infty \xrightarrow{(-)^{\mathrm{op}}} \mathrm{Cat}_\infty,$$

which immediately implies the claim, since taking the opposite category is an involution, i.e. an equivalence of categories, and so commutes with limits. □

Definition 2.7. For an $\mathbb{E}_\infty$-ring R we refer to

$$\mathrm{CAlg}_R := \mathrm{CAlg}(\mathrm{Mod}_R),$$

$$\mathrm{cCAlg}_R := \mathrm{cCAlg}(\mathrm{Mod}_R)$$

as the category of R-algebras and R-coalgebras, respectively. Moreover, we write $\mathrm{CAlg}_R^{\mathrm{cn}}$ and $\mathrm{cCAlg}_R^{\mathrm{cn}}$ for the full subcategory spanned by the (co)-algebras whose underlying R-module is connective.

Remark 2.8. Let $f : R \to R'$ be a morphism of commutative ring spectra, then we have the well known adjunction between base change and restriction of scalars

$$\mathrm{Mod}_R \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Mod}_{R'} \ .$$

The functor f^* is strong symmetric monoidal, while f_* is lax symmetric monoidal. Hence, they induce an adjunction fitting in the commutative diagram

$$\begin{array}{ccc} \mathrm{CAlg}_R & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathrm{CAlg}_{R'} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_R & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathrm{Mod}_{R'} \end{array} .$$

The functor f_* is however *not oplax symmetric monoidal* and so does not induce a functor on coalgebras. Nonetheless, since colimits of coalgebras are formed underlying and $f^* : \mathrm{Mod}_R \to \mathrm{Mod}_{R'}$ commutes with colimits, so does $f^* : \mathrm{cCAlg}_R \to \mathrm{cCAlg}_{R'}$. Hence, by the adjoint functor theorem, the base change functor on coalgebras does admit a right adjoint, which we denote $f_!$. Notice that the diagram

$$\begin{array}{ccc} \mathrm{cCAlg}_R & \xleftarrow{f_!} & \mathrm{cCAlg}_{R'} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_R & \xleftarrow{f_*} & \mathrm{Mod}_{R'} \end{array}$$

does *not commute*. Indeed, since R' and R are the terminal objects in $\mathrm{cCAlg}_{R'}$ and cCAlg_R respectively, we get that $f_!(R') = R$ as $f_!$ preserves limits. We do not know of a general formula for $f_!$, however we will see that it plays a central role in formulating the deformation theory of coalgebras.

2.2. **Presentably symmetric monoidal ∞-categories and duality.** We now restrict our attention to coalgebras in *presentably symmetric monoidal* ∞-categories. We have already seen in Proposition 2.5 that such coalgebra categories are again presentable, which is essential for the existence of the right adjoint $f_! : \mathrm{cCAlg}_{R'} \to \mathrm{cCAlg}_R$ discussed in Remark 2.8. In a presentably symmetric monoidal ∞-category we also have a well behaved notion of *duality* between coalgebras and algebras, which we wil discuss next.

Definition 2.9. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories and functors which commute with colimits. By [Lur17, Proposition 4.8.1.15.] $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ inherits a natural symmetric monoidal structure, such that we have a map $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ exhibiting $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ as the category of functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which commute with colimits in each variable separately. We call $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ the category of *presentably symmetric monoidal ∞-categories*.

Example 2.10. If R is an $\mathbb{E}_\infty$-ring then the category of R-module spectra Mod_R is presentably symmetric monoidal.

Remark 2.11. If $\mathcal{C}$ is presentably symmetric monoidal, the functor $-\otimes-$ commutes with colimits in both variables separately. Thus, since $\mathcal{C}$ is presentable, for every $X \in \mathcal{C}$ the functor $X \otimes - : \mathcal{C} \to \mathcal{C}$ admits a right adjoint, i.e. $\mathcal{C}$ is also closed monoidal. We denote this right adjoint by $\mathrm{map}_{\mathcal{C}}(X, -)$. This assignment defines a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$, and we denote its adjoint as

$$\mathrm{map}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}.$$

Notice that, since $-\otimes-$ is symmetric, we have that

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(Y, \mathrm{map}_{\mathcal{C}}(X, Z)) &\simeq \mathrm{Map}_{\mathcal{C}}(X \otimes Y, Z)\\ &\simeq \mathrm{Map}_{\mathcal{C}}(X, \mathrm{map}_{\mathcal{C}}(Y, Z))\\ &\simeq \mathrm{Map}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{map}_{\mathcal{C}}(Y, Z), X)\end{aligned}$$

for all $X, Y, Z \in \mathcal{C}$. Thus, for each $Z \in \mathcal{C}$, the functor $\mathrm{map}_{\mathcal{C}}(-, Z)$ is adjoint to itself.

For a presentably symmetric monoidal ∞-category $\mathcal{C}$ there is also a "generalized suspension coalgebra" functor $\mathcal{S} \to \mathrm{cCAlg}(\mathcal{C})$ which we now construct. In particular this gives the coalgebra structure on the R-module $C_*(X; R)$ for R any $\mathbb{E}_\infty$-ring and X a space.

Proposition 2.12. *Let $\mathcal{C}$ a presentably symmetric monoidal ∞-category. Then the functor*

$$\mathcal{S} \to \mathcal{C} \qquad X \mapsto 1_{\mathcal{C}}[X]$$

which sends a space X to the colimit over the constant diagram $X \to * \xrightarrow{1_{\mathcal{C}}} \mathcal{C}$ is symmetric monoidal with respect to the cartesian monoidal structure on $\mathcal{S}$.

Proof. Since $-\otimes-$ commutes with colimits in both variables separably by assumption, we have that:

$$(\operatorname*{colim}_X 1_{\mathcal{C}}) \otimes (\operatorname*{colim}_Y 1_{\mathcal{C}}) \simeq \operatorname*{colim}_X \operatorname*{colim}_Y (\underbrace{1_{\mathcal{C}} \otimes 1_{\mathcal{C}}}_{\simeq 1_{\mathcal{C}}}) \simeq \operatorname*{colim}_{X\times Y} 1_{\mathcal{C}}.$$

□

Lemma 2.13. *Suppose $\mathcal{C}$ is an ∞-category which admits finite products and equip it with the cartesian monoidal structure. Then the forgetful functor $\mathrm{cCAlg}(\mathcal{C}) \to \mathcal{C}$ is an equivalence, with inverse given by equipping an object $X \in \mathcal{C}$ with the comultiplication given by the diagonal map $X \to X \times X$ and counit given by the terminal map $X \to *_{\mathcal{C}}$.*

Proof. By [Lur17, Corollary 2.4.4.10.] the map $\mathrm{CAlg}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{C}^{\mathrm{op}}$ is an equivalence, hence the claim follows by applying $(-)^{\mathrm{op}}$. □

Example 2.14. By Proposition 2.12, for each $\mathbb{E}_\infty$-ring R the singular chains functor

$$\begin{aligned}\mathcal{S} &\to \mathrm{Mod}_R \\ X &\mapsto R[X]\end{aligned}$$

is strong symmetric monoidal and thus induces a functor

$$R[-] : \mathcal{S} \simeq \mathrm{cCAlg}(\mathcal{S}) \to \mathrm{cCAlg}_R.$$

Hence, the R-homology of a space X carries a natural coalgebra structure.

Construction 2.15. Suppose $\mathcal{C}$ is a closed symmetric monoidal category, then we have a natural map

$$\mathrm{map}_{\mathcal{C}}(A, B) \otimes \mathrm{map}_{\mathcal{C}}(A, B) \to \mathrm{map}_{\mathcal{C}}(A \otimes A, B \otimes B)$$

which is adjoint to the double evaluation

$$(A \otimes \mathrm{map}_{\mathcal{C}}(A, B)) \otimes (A \otimes \mathrm{map}_{\mathcal{C}}(A, B)) \xrightarrow{\mathrm{ev}\otimes\mathrm{ev}} B \otimes B.$$

This can be phrased elegantly by saying that the functor

$$\mathrm{map}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

is lax monoidal with respect to the monoidal structure on $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, defined by

$$(A, B) \otimes_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} (C, D) := (A \otimes C, B \otimes D).$$

Thus, it induces a functor

$$\mathrm{map}_{\mathcal{C}}(-, -) : \mathrm{CAlg}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \simeq \mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}} \times \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}),$$

so in particular, for each $R \in \mathrm{CAlg}(\mathcal{C})$ we get a functor

$$\mathrm{map}_{\mathcal{C}}(-, R) : \mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}} = \mathrm{CAlg}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{CAlg}(\mathcal{C}).$$

For a coalgebra $A \in \mathrm{cCAlg}(\mathcal{C})$ we call the algebra $\mathrm{map}_{\mathcal{C}}(A, 1_{\mathcal{C}})$ the *dual algebra* of A.

Example 2.16. As a special case, we see that for every $\mathbb{E}_\infty$-ring R and every space X the R-cohomology of X

$$\mathrm{map}_{\mathrm{Sp}}(\mathbb{S}[X], R) \simeq \mathrm{map}_R(R[X], R) \simeq \lim_X R = R^X$$

inherits a ring structure from the coalgebra structure on the R-homology $R[X]$.

Proposition 2.17. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and denote by $\mathcal{C}^{\mathrm{dual}}$ the full subcategory of dualizable objects. Then the assignment $X \mapsto X^\vee$ induces a strong monoidal equivalence $(\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}} \simeq \mathcal{C}^{\mathrm{dual}}$. Moreover, if $\mathcal{C}$ is closed monoidal, then the dual is given by $X^\vee \simeq \mathrm{map}_{\mathcal{C}}(X, 1_{\mathcal{C}})$.*

Proof. This is [Lur18, Proposition 3.2.4]. □

Corollary 2.18. *For every symmetric monoidal ∞-category $\mathcal{C}$ the functor*

$$\mathrm{cCAlg}(\mathcal{C}^{\mathrm{perf}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{C}^{\mathrm{perf}})$$
$$A \mapsto A^{\vee}$$

is an equivalence of categories with inverse taking $R \in \mathrm{CAlg}(\mathcal{C}^{\mathrm{perf}})$ to the dual $R^{\vee}$ with the induced coalgebra structure.

Remark 2.19. Observe that, since the tensor product of two dualizable objects $X, Y \in \mathcal{C}$ is again dualizable with $(X \otimes Y)^{\vee} \simeq X^{\vee} \otimes Y^{\vee}$, the inclusion functor $\mathcal{C}^{\mathrm{perf}} \hookrightarrow \mathcal{C}$ exhibits $\mathrm{cCAlg}(\mathcal{C}^{\mathrm{perf}})$ as the full subcategory of $\mathrm{cCAlg}(\mathcal{C})$ spanned by those coalgebras whose underlying object is dualizable. We call a coalgebra $A \in \mathrm{cCAlg}(\mathcal{C})$ *dualizable* if it belongs to this category.

For a symmetric monoidal ∞-category $\mathcal{C}$ and $X, Y \in \mathcal{C}$ with Y, we have that by definition the space of maps $X \to Y$ is equivalent to the space of maps $X^{\vee} \to Y^{\vee}$. The corresponding statement for maps of coalgebras holds as well, however in the derived setting this is not immediately clear. To this end we prove a lemma, during which employ the following terminology:
For a functor $F : \mathcal{C} \to \mathcal{D}$ we say that $A \in \mathcal{C}$ is F-local, if the natural transformation $\mathrm{Map}_{\mathcal{C}}(A, -) \to \mathrm{Map}_{\mathcal{D}}(F(A), F(-))$ is an equivalence.

Lemma 2.20. *Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor. Suppose we have $R \in \mathrm{CAlg}(\mathcal{C})$ such that each tensor power $R^{\otimes n}$ considered as an object in $\mathcal{C}$ is F-local. Then the map*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(R, S) \xrightarrow{F} \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D})}(F(R), F(S))$$

is an equivalence.

Proof. Since F is lax symmetric monoidal, it induces a map of ∞-operads

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{\quad f \quad} & \mathcal{D}^{\otimes} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & \mathrm{Fin}_* & \end{array}$$

which takes any $(A_1, \dots, A_n) \in \mathcal{C}^{\otimes}_{\langle n \rangle}$ to the point $(F(A_1), \dots, F(A_n)) \in \mathcal{D}^{\otimes}_{\langle n \rangle}$. A commutative algebra structure on an object $R \in \mathcal{C}$ is precisely given by a a section s_R of p which takes $\langle n \rangle$ to $(R, \dots, R) \in \mathcal{C}_{\langle n \rangle}$ and maps inert morphisms to inert morphisms. Let $\varphi : \langle n \rangle \to \langle m \rangle$ be a morphism in Fin_* and denote by $p_i : \langle n \rangle \to \langle 1 \rangle$ the unique inert map which sends $i \mapsto 1$. For each i we have a factorization

$$\langle n \rangle \xrightarrow{\psi_i} \langle k_i \rangle \xrightarrow{\pi_i} \langle 1 \rangle$$

of $p_i \circ \varphi$ into an inert map ψ_i and an active map π_i. Then for each $B = (B_1, \dots B_m) \in \mathcal{C}^{\otimes}$ we get equivalences

$$\begin{aligned}
\mathrm{Map}^{\varphi}_{\mathcal{C}^{\otimes}}(s_R(\langle n\rangle), B) &\simeq \prod_{i=1,\dots,m} \mathrm{Map}^{p_i\circ\varphi}_{\mathcal{C}^{\otimes}}((R,\dots,R), B_i) \\
&\simeq \prod_{i=1,\dots,m} \mathrm{Map}_{\mathcal{C}}(R^{\otimes k_i}, B_i) \\
&\simeq \prod_{i=1,\dots,m} \mathrm{Map}_{\mathcal{D}}(F(R)^{\otimes k_i}, F(B_i)) \\
&\simeq \prod_{i=1,\dots m} \mathrm{Map}^{p_i\circ\varphi}_{\mathcal{D}^{\otimes}}((f\circ s_R)(\langle n\rangle), B_i) \\
&\simeq \mathrm{Map}^{\varphi}_{\mathcal{D}^{\otimes}}((f\circ s_R)(\langle n\rangle), B),
\end{aligned}$$

hence each value of $s_R : \mathrm{Fin}_* \to \mathcal{C}^{\otimes}$ is f-local, and thus s_R is local with respect to the functor $f_* : \Gamma(p) \to \Gamma(q)$. Since $\mathrm{CAlg}(\mathcal{C})$ and $\mathrm{CAlg}(\mathcal{D})$ are full subcategories of $\Gamma(p)$ and $\Gamma(q)$ respectively, the claim follows. ☐

Proposition 2.21. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $A, B \in \mathrm{cCAlg}(\mathcal{C})$ with A dualizable. Then the natural map*

$$\mathrm{Map}_{\mathrm{cCAlg}(\mathcal{C})}(B, A) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A^\vee, B^\vee)$$

is a homotopy equivalence.

Proof. Apply Lemma 2.20 to the duality functor $(-)^\vee : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. ☐

Because algebras are much better understood than coalgebras, especially in ∞-land, we will rely on this proposition to deduce as much as possible about the coalgebraic setting from already established results.

3. Abstract deformation theory

In this chapter we review the theory of square zero extensions and the deformation theory of functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, as developed in [Lur17] and [Lur12].
In the following let R denote a connective $\mathbb{E}_\infty$-ring and $\widetilde{R} \to R$ a square zero extension with fiber $M \in \mathrm{Mod}^{\mathrm{cn}}_R$. The goal of deformation theory is to describe the fiber $\mathrm{fib}_A(X(\widetilde{R}) \to X(R))$ over some point $A \in X(R)$, that is we want to understand the different ways A can be lifted to a point of $X(\widetilde{R})$. Problems like these already abound in ordinary mathematics, however the derived setting allows for significantly more flexibility and conceptional clarity, sidestepping the need for ad-hoc introduction of cohomology groups and cocycle computations. The approach due to Lurie is as follows: Square zero extensions $\widetilde{R} \to R$ with fiber M are obtained by pulling back along maps into a classifying object $R \oplus M[1] \in \mathrm{cCAlg}_{/R}$, namely the split square zero extension with fiber $M[1]$. For well behaved functors

$$X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$$

the space $X(\widetilde{R})$ can then also be obtained by pulling back along the induced map $X(R) \to X(R \oplus M[1])$. In this case the fiber $\mathrm{fib}_A(X(\widetilde{R}) \to X(R))$ is given a path space in $X(R \oplus M[1])$. Moreover, for each $A \in X(R)$ writing

$$X_A^{R\oplus M[n]} := \mathrm{fib}_A(X(R \oplus M[n]) \to X(R))$$

the sequence $\{X_A^{R\oplus M[n]}\}_{n\in\mathbb{N}}$ defines a spectrum $T_{X_A}^M$ which thus combines all of the information about lifting against square zero extensions with fiber $M[n]$ for some n into one spectrum. For $M = R$ this is called the *tangent complex* of X at the point A. The major inconvenience here is that in general $T_{X_A}^M$ is not determined by T_{X_A}, and so lifting against a square zero extension with fiber M may be a very different problem than lifting against one with fiber R. In some cases however, the functor

$$\mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S} \qquad M \mapsto X_A^{R\oplus M}$$

is almost corepresentable, i.e. given by $\mathrm{Map}_R(L_{X_A}, -)$ for some not necessarily connective R-Module L_{X_A} called the *cotangent complex* of X at the point A. We can think of this as X being "infinitesimally representable". For an in-depth exploration of the question of how far this is from being actually representable see [Lur12]. For our purposes, this effectively reduces the problem of describing lifts of $A \in X(R)$ along an arbitrary square zero extension of R to the computation of a single spectrum. For an $\mathbb{E}_\infty$-ring S and the functor

$$\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \qquad R \mapsto X(R) = \mathrm{Spec}(S)(R)$$

the spectrum L_{X_R} is closely related to the usual cotangent complex L_S of the ring S, hence the terminology.

3.1. **Square zero extensions and deformations.**

Proposition 3.1. *For every $\mathbb{E}_\infty$-ring R there is an equivalence of categories*

$$\mathrm{Mod}_R \xrightarrow{\sim} \mathrm{Sp}(\mathrm{CAlg}_{/R})$$

such that for each $n \geq 0$ the functor

$$\mathrm{Mod}_R \xrightarrow{\sim} \mathrm{Sp}(\mathrm{CAlg}_{/R}) \xrightarrow{\Omega^{\infty-n}} \mathrm{CAlg}_{/R}$$

sends a module M to an augmented R-algebra whose underlying R-module is given by the direct sum $R \oplus M[n]$.

Proof. This is [Lur17, Theorem 7.3.4.13]. □

Remark 3.2. For a connective $\mathbb{E}_\infty$-ring R and a connective R-module M we call $\Omega^\infty M = R \oplus M$ with the R-algebra structure described above the *split square zero extension* of R along M. If R and M are discrete, the multiplication is explicitly given by

$$(a + m)(b + n) := ab + an + mb.$$

In particular, in this case we have that $R \oplus R \simeq R[x]/x^2$.

Definition 3.3. For a connective $\mathbb{E}_\infty$-ring R and a connective R-module M we say that a map $R^\eta \to R$ is a *square zero extension* of R along M if it fits into a pullback diagram

$$\begin{array}{ccc} R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ R & \xrightarrow[(\mathrm{id},\eta)]{} & R \oplus M[1]. \end{array}$$

Moreover, we call the mapping space $\mathrm{Map}_{\mathrm{CAlg}_{/R}}(R, R \oplus M)$ the space of *derivations* $\eta : R \to M$.

This definition make sense without any connectivity assumptions. However, since our interest lies primarily in the case where everything is connective, we have chosen to include them in the definition to avoid awkward terminology.

Remark 3.4. Note that the split square zero extension is precisely the one classified by the zero derivation

$$R \xrightarrow{(\mathrm{id},0)} R \oplus M[1].$$

Moreover, if $R^\eta \to R$ is any square zero extension then by taking fibers we get a commutative diagram with exact columns

$$\begin{array}{ccc} M & \xrightarrow{\sim} & M \\ \downarrow & & \downarrow{\scriptstyle 0} \\ R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ R & \xrightarrow{(\mathrm{id},\eta)} & R \oplus M[1]. \end{array}$$

So, in particular, we have a fiber sequence $M \to R^\eta \to R$. Moreover, by applying $- \otimes_R M$ to the diagram we get

$$\begin{array}{ccc} M^{\otimes 2} & \xrightarrow{\sim} & M^{\otimes 2} \\ \downarrow & & \downarrow{\scriptstyle 0} \\ R^\eta \otimes_R M & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ M & \longrightarrow & M \oplus M^{\otimes 2}[1]. \end{array}$$

Hence, the composition $M^{\otimes 2} \to R^\eta \otimes_R M \to M$ is nullhomotopic, i.e. the action of M on itself induced from the R^η-action is trivial. This is the sense in which these extensions are "square zero".

Remark 3.5. For any connective R-module M the augmented R-algebra $\Omega^\infty M = R \oplus M$ inherits the structure of an $\mathbb{E}_\infty$-monoid in $\mathrm{CAlg}_{/R}$ with delooping given by $R \oplus M[1]$. Thus, we can think of $R \oplus M[1]$ as the classifying object for square zero extensions with fiber M, the "universal" derivation being given by the trivial section $R \to R \oplus M[1]$. From this perspective, a square zero extension is precisely a $R \oplus M$-torsor in $\mathrm{CAlg}_{/R}$.

Defining square zero extensions in this abstract, very general way will be very useful for our deformation theoretic purposes. However, we also need a criterion for deciding whether a given map of $\mathbb{E}_\infty$-rings is a square zero extension. This is provided by the following statement.

Proposition 3.6. *Let $R' \to R$ be a map of connective $\mathbb{E}_\infty$-rings with fiber M such that $M \in \mathrm{Sp}_{\geq n} \cap \mathrm{Sp}_{\leq 2n}$ and the multiplication map $M \otimes_{R'} M \to M$ is nullhomotopic. Then $R' \to R$ is a square zero extension.*

Proof. This is immediate from [Lur17, Theorem 7.4.1.23.]. □

Example 3.7. If $R' \to R$ is a surjective map of ordinary commutative rings with kernel $M \subseteq R'$, then $R' \to R$ is a square zero extension if and only if $M^2 = 0$. In particular, for every n the map $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1}$ is square zero with kernel $\mathbb{F}_p$.

Example 3.8. If R is any $\mathbb{E}_\infty$-ring, then the map $\tau_{\leq n} R \to \tau_{\leq n-1} R$ is a square zero extension with fiber $\pi_n R[n]$.

For a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and a square zero extension $R^\eta \to R$ we want to study the fibers of the map $X(R^\eta) \to X(R)$, i.e. given $A \in X(R)$ we wish to understand the space of deformations of A to an object $\widetilde{A} \in X(R^\eta)$. Notice that, since R and M are connective, the derivation

$$R \xrightarrow{(0,\eta)} R \oplus M[1]$$

is surjective on π_0, indeed an isomorphism. This motivates the following definition for a class of functors which is "well behaved" with regard to square zero extensions.

Definition 3.9. A functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is called:

(1) *Cohesive* if for any pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

which induces surjections $\pi_0 R \to \pi_0 S$ and $\pi_0 S' \to \pi_0 S$, the diagram

$$\begin{array}{ccc} X(R') & \longrightarrow & X(S') \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(S) \end{array}$$

is a pullback of spaces. We refer to such pullbacks of $\mathbb{E}_\infty$-rings as *small pullbacks*.

(2) *Nilcomplete* if for every connective $\mathbb{E}_\infty$-ring R the natural map

$$X(R) \to \varprojlim_n X(\tau_{\leq n} R)$$

is a homotopy equivalence.

Example 3.10. Let R be a connective $\mathbb{E}_\infty$-ring, then the functor

$$\mathrm{Spec}(R) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \qquad S \mapsto \mathrm{Map}_{\mathrm{CAlg}}(R, S)$$

is cohesive since the inclusion $\mathrm{CAlg}^{\mathrm{cn}} \hookrightarrow \mathrm{CAlg}$ commutes with small pullbacks and $\mathrm{Spec}(R)$ is the restriction of the corepresented functor $\mathrm{Map}_{\mathrm{CAlg}}(R, -) : \mathrm{CAlg} \to \mathcal{S}$.

Construction 3.11. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a cohesive functor, $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and $A \in X(R)$. Then we define a functor

$$\mathrm{X}_A^- : \mathrm{CAlg}^{\mathrm{cn}}_{/R} \to \mathcal{S} \qquad (R' \to R) \mapsto \mathrm{fib}_A(X(R') \to X(R)).$$

We call $\mathrm{X}_A^{R'}$ the *space of deformations* of A along $R' \to R$.

Remark 3.12. If X is cohesive and nilcomplete, then for every connective $\mathbb{E}_\infty$-ring R and every $A \in X(\tau_{\leq 0} R)$ we have an equivalence

$$X_A^R \simeq \varprojlim_n X_A^{\tau_{\leq n} R},$$

i.e. we can construct lifts to R by inductively lifting against the square zero extensions $\tau_{\leq n} R \to \tau_{\leq n-1} R$. This will be very useful for applications, however we will not need nilcompleteness for the theoretical groundwork that makes up the rest of this chapter.

3.2. **The tangent and cotangent complex.**

Definition 3.13. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ is called:

(1) *Reduced* if it preserves the terminal object.
(2) *Excisive* if it takes pushouts to pullbacks.

Proposition 3.14. *Let $X : \mathrm{CAlg} \to \mathcal{S}$ be cohesive and $A \in X(R)$ be an R-valued point. Then the functor given by the composition*

$$\mathrm{Mod}_R^{\mathrm{cn}} \xrightarrow{\Omega^\infty|_{\mathrm{Mod}_R^{\mathrm{cn}}}} \mathrm{CAlg}^{\mathrm{cn}}_{/R} \xrightarrow{\mathrm{X}_A^-} \mathcal{S}$$

is reduced and excisive.

Proof. Clearly, we have $\mathrm{X}_A^R \simeq *$, so the functor is reduced. Since X is cohesive and taking fibers commutes with limits, the functor X_A^- takes small pullbacks to pullbacks. Hence, the claim follows from [Lur17, Proposition 1.4.2.13] by observing that $\Omega^\infty|_{\mathrm{Mod}_R^{\mathrm{cn}}}$ sends

$$\begin{array}{ccc} M & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M[1] \end{array}$$

to the small pullback

$$\begin{array}{ccc} R \oplus M & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ R & \xrightarrow[(\mathrm{id},0)]{} & R \oplus M[1], \end{array}$$

i.e. we have $\Omega \mathrm{X}_A^{R\oplus M[1]} \simeq \mathrm{X}_A^{R\oplus M}$ for any $M \in \mathrm{Mod}_R^{\mathrm{cn}}$. □

Construction 3.15. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be cohesive and $A \in X(R)$. Then by [Lur17, Proposition 1.4.2.22] we obtain an essentially unique factorization

$$\begin{array}{ccc} & & \mathrm{Sp} \\ & {\scriptstyle T^{-}_{X_A}} \nearrow & \downarrow {\scriptstyle \Omega^\infty} \\ \mathrm{Mod}_R^{\mathrm{cn}} & \xrightarrow[X_A^{R\oplus -}]{} & \mathcal{S} \end{array}$$

where for $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ the spectrum $T^M_{X_A}$ is given by the sequence of spaces $\{\mathrm{X}_A^{R\oplus M[n]}\}_n$. For $M = R$ we call $T^R_{X_A} =: T_{X_A}$ the *tangent complex* of A.

Warning 3.16. The name tangent complex is a historical convention and somewhat misleading. In general, the spectrum T_{X_A} is not contained in the full subcategory $D(\mathbb{Z}) \subseteq \mathrm{Sp}$ i.e. cannot be modeled by a chain complex of abelian groups.

Lemma 3.17. *For a connective $\mathbb{E}_\infty$-ring R denote by $\mathrm{Mod}_R^{\mathrm{acn}} \subseteq \mathrm{Mod}_R$ the full subcategory spanned by those R-modules which are contained in $(\mathrm{Mod}_R)_{\geq n}$ for some n and let $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ be an excisive functor. Then F admits an extension to an excisive functor $\mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$ which is unique up to contractible choice.*

Proof. This is [Lur12, Lemma 1.3.2]. □

Proposition 3.18. *Let R be an $\mathbb{E}_\infty$-ring, $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be cohesive and $A \in X(R)$. Then T_{X_A} inherits a natural R-module structure such that for any perfect connective R-module M we have a natural equivalence $T^M_{X_A} \simeq T_{X_A} \otimes_R M$.*

Proof. By Lemma 3.17 we can extend the functor $F = X_{X_A}^{R\oplus -} : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ uniquely to an excisive functor $F' : \mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$. Since $\mathrm{Mod}_R^{\mathrm{acn}}$ is stable, F' is an exact functor. Thus, the restriction $F'|_{\mathrm{Mod}_R^{\mathrm{perf}}}$ is also exact, hence, since Sp is stable, [Lur17, Proposition 1.4.2.22] implies that we get an essentially unique lift to an exact functor $\widetilde{F} : \mathrm{Mod}_R^{\mathrm{perf}} \to \mathrm{Sp}$. Finally, applying [Lur08, Proposition 5.5.1.9] we see that $\widetilde{F}$ induces a colimit preserving functor $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{perf}}) \simeq \mathrm{Mod}_R \to \mathrm{Sp}$, which under the equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Mod}_R, \mathrm{Sp}) \simeq \mathrm{Mod}_R \qquad G \mapsto G(R)$$

yields a R-module whose underlying spectrum is given by T_{X_A}. □

Proposition 3.19. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a cohesive functor and $R^\eta \to R$ a square zero extension classified by a derivation $R \xrightarrow{\eta} M[1]$. Then for each $A \in X(R)$ the space of deformations $\mathrm{X}_A^{R^\eta}$ is either empty or a torsor under the grouplike $\mathbb{E}_\infty$-monoid $\Omega^\infty T^M_{X_A}$. Moreover, η determines an obstruction class in $\pi_{-1} T^M_{X_A}$, which vanishes if and only if $\mathrm{X}_A^{R^\eta}$ is non-empty.*

Proof. Since X is cohesive, applying X_A^- to the pullback diagram defining R^η

$$\begin{array}{ccc} R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle 0} \\ R & \xrightarrow[(0,\eta)]{} & R \oplus M[1], \end{array}$$

we get a pullback of spaces

$$\begin{array}{ccc} \mathrm{X}_A^{R^\eta} & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle A^0} \\ * & \xrightarrow[A^\eta]{} & \mathrm{X}_A^{R\oplus M[1]}, \end{array}$$

exhibiting $\mathrm{X}_A^{R^\eta}$ as the space of paths in $\mathrm{X}_A^{R\oplus M[1]}$ between the points A^0 and A^η. Hence, it is non-empty if and only if the homotopy class determined by the map

$$* \xrightarrow{A^\eta} \mathrm{X}_A^{R\oplus M[1]} \simeq \Omega^\infty T_{X_A}^M[1]$$

vanishes. Moreover, in this case $\mathrm{X}_A^{R^\eta}$ is a torsor under the loop space based at A^0, which is given by

$$\Omega \mathrm{X}_A^{R\oplus M[1]} \simeq \mathrm{X}_A^{R\oplus M} \simeq \Omega^\infty T_{X_A}^M.$$

□

Proposition 3.20. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be cohesive and $R \to R'$ a map of connective $\mathbb{E}_\infty$-rings. Moreover, let $A \in X(R)$ be a R-valued point and denote by A' the image of A under the induced map $X(R) \to X(R')$. Then for every $M \in \mathrm{Mod}_{R'}^{\mathrm{cn}}$ we have a natural map*

$$T_{X_A}^M \to T_{X_{A'}}^M$$

which is an equivalence if the map $\pi_0 R \to \pi_0 R'$ is surjective.

Proof. Applying X to the pullback of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R \oplus M & \longrightarrow & R' \oplus M \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

and taking the fibers over the points $A \in X(R)$ and $A' \in X(R')$ gives a commutative diagram

$$\begin{array}{ccc} X_A^{R\oplus M} & \longrightarrow & X_{A'}^{R'\oplus M} \\ \downarrow & & \downarrow \\ X(R\oplus M) & \longrightarrow & X(R'\oplus M) \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(R') \end{array}.$$

The map $X_A^{R\oplus M} \to X_{A'}^{R'\oplus M}$ is natural in M and thus gives a map of spectra $T^M_{X_A} \to T^M_{X_{A'}}$ as claimed. Moreover, if $R \to R'$ is surjective on π_0, then the pullback of $\mathbb{E}_\infty$-rings is small. Hence, since X is cohesive, the map $X_A^{R\oplus M} \to X_{A'}^{R'\oplus M}$ is an equivalence and thus the induced map $T^M_{X_A} \to T^M_{X_{A'}}$ is as well. ☐

Notice that, if $R^\eta \to R$ is a square zero extension of connective $\mathbb{E}_\infty$-rings the map $\pi_0 R^\eta \to \pi_0 R$ is necessarily surjective. Thus, if we are given $A \in X(R)$ and a lift $A^\eta \in X(R^\eta)$, we know that if we have $M \in \mathrm{Mod}^{\mathrm{cn}}_{R^\eta}$ such that the R^η-action factors through R, then $T^M_{X_{A^\eta}}$ agrees with $T^M_{X_A}$. The following Proposition shows that we can compute the value of $T^-_{X_{A^\eta}}$ on arbitrary connective R^η-modules in terms of $T^-_{X_A}$.

Proposition 3.21. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be cohesive, $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and $A' \in X(R)$. Let $R^\eta \to R$ be a square zero extension with fiber M such that A' admits a lift $A \in X(R^\eta)$. Then for any $N \in \mathrm{Mod}^{\mathrm{cn}}_{R^\eta}$ if we have that $T^{M\otimes_R(R\otimes_{R^\eta} N)}_{X_{A'}} \simeq T^{N\otimes_{R^\eta} R}_{X_{A'}} \simeq 0$ it follows that $T^N_{X_A} \simeq 0$.*

Proof. Applying the functor $- \otimes_{R^\eta} N$ to the extension

$$M \to R^\eta \to R$$

yields a cofiber sequence

$$M \otimes_{R^\eta} N \to N \to N \otimes_{R^\eta} R$$

of connective R^η-modules. Now the functor $T^-_{X_A} : \mathrm{Mod}^{\mathrm{cn}}_{R^\eta} \to \mathrm{Sp}$ is excisive, hence we get a fiber sequence of spectra

$$T^{M\otimes_{R^\eta} N}_{X_A} \to T^N_{X_A} \to T^{N\otimes_{R^\eta} R}_{X_A}.$$

Since the extension $R^\eta \to R$ is square zero, the action of R^η on M factors through R, i.e. we have that

$$M \otimes_{R^\eta} N \simeq (M \otimes_R R) \otimes_{k^\eta} N \simeq M \otimes_R (R \otimes_{R^\eta} N).$$

Applying Proposition 3.20 we see that

$$T^{M\otimes_R(R\otimes_{R^\eta} N)}_{X_A} \simeq T^{M\otimes_R(R\otimes_{R^\eta} N)}_{X_{A'}} \simeq 0$$

and similarly

$$T^{N\otimes_{R^\eta} R}_{X_A} \simeq T^{N\otimes_{R^\eta} R}_{X_{A'}} \simeq 0,$$

which proves the claim. ☐

Remark 3.22. Note that in the setting of Proposition 3.21, if M and $N \otimes_{R^\eta} R$ are perfect R-modules, Proposition 3.18 implies that it suffices to assume that $T_{X_{A'}} \simeq 0$. Moreover, as part of the proof we have seen that every connective R^η-Module N sits in a cofiber sequence

$$M \otimes_R (R \otimes_{R^\eta} N) \to N \to R \otimes_{R^\eta} N.$$

If we think of N as a lift of $R\otimes_{R^\eta} N$ along the square zero extension $R^\eta \to R$, this is part of a description of the deformation theory of connective modules. The complete description may be deduced from Proposition 4.1.

Definition 3.23. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be cohesive, we say that X *admits a cotangent complex* at a point $\varphi \in X(R)$ if there exists a (not necessarily connective) R-module L_{X_φ}, together with for every $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ an equivalence

$$\mathrm{Map}_R(L_{X_\varphi}, M) \simeq \mathrm{fib}_\varphi(X(R \oplus M) \to X(R)) = X_\varphi^{R\oplus M},$$

which is natural in M.

Remark 3.24. Observe that, if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a cotangent complex at a point $\varphi \in X(R)$, then for every connective R-module M, considering the composition

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{M} \mathrm{Mod}_R \underset{\mathrm{Map}_R(L_{X_\varphi},-)}{\overset{X_\varphi^{R\oplus -}}{\underset{\Downarrow}{\rightrightarrows}}} \mathcal{S}$$

gives a natural equivalence of R-modules

$$\mathrm{map}_R(L_{X_\varphi}, M) \simeq T_{X_\varphi}^M.$$

In this sense, the tangent complex is the dual of the cotangent complex. The cotangent complex is a much more useful invariant, since it does not depend on the module M. However, as we will see in Example 3.26, not every cohesive functor admits a cotangent complex.

As an example we now discuss how the well-known cotangent complex of an $\mathbb{E}_\infty$-ring R relates to this formalism. Later we will use this comparison to analyze the deformation theory of dualizable coalgebras.

Example 3.25. Let R be any $\mathbb{E}_\infty$-ring. The composition

$$\mathrm{Mod}_R \xrightarrow{\Omega^\infty} \mathrm{CAlg}_{/R} \xrightarrow{\mathrm{Map}(R,-)} \mathcal{S}$$

is accessible and commutes with limits. Thus, since Mod_R is presentable, the adjoint functor theorem implies that it is corepresented by an R-Module L_R called the *cotangent complex* of R. Now the functor

$$X = \mathrm{Spec}(R) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \qquad S \mapsto \mathrm{Map}_{\mathrm{CAlg}}(R, S)$$

is clearly cohesive. Moreover, for any $(\varphi : R \to S) \in \mathrm{Spec}(R)(S)$ and $M \in \mathrm{Mod}_S^{\mathrm{cn}}$ we get that

$$\begin{aligned} &\mathrm{fib}_\varphi(\mathrm{Map}_{\mathrm{CAlg}}(R, S \oplus M) \to \mathrm{Map}_{\mathrm{CAlg}}(R, S)) \\ \simeq\ &\mathrm{Map}_{\mathrm{CAlg}_{/S}}(R, S \oplus M) \\ \simeq\ &\mathrm{Map}_{\mathrm{CAlg}_{/R}}(R, R \oplus \varphi_* M) \\ \simeq\ &\mathrm{Map}_R(L_R, \varphi_* M) \\ \simeq\ &\mathrm{Map}_S(\varphi^* L_R, M). \end{aligned}$$

Hence, we see that the functor $\mathrm{Spec}(R)$ admits a cotangent complex at every point $\varphi \in \mathrm{Spec}(R)(S)$, given by the S-module $\varphi^* L_R$. Explicitly, this tells us that the space of lifts in the diagram

$$\begin{array}{ccc} & & S \oplus M \\ & \nearrow & \downarrow \\ R & \xrightarrow{\varphi} & S \end{array}$$

is naturally identified with $\Omega^\infty T^M_{X_\varphi} = \mathrm{Map}_S(\varphi^* L_R, M)$. Moreover, if $L_R \simeq 0$, then R admits unique lifts against *arbitrary* square zero extensions. In the case $S = R$ and $\varphi = \mathrm{id}$ $\mathrm{Map}_{\mathrm{CAlg}_{/R}}(R, R \oplus M) = \mathrm{Map}_R(L_R, M)$ is also called the space of *derivations* $R \to M$, which in the discrete case can be explicitly described via additive maps satisfying the Leibniz rule. Although the existence of a cotangent complex for coalgebras remains unclear, we will show that there is a coalgebraic notion of derivations which play a similar role in the deformation theory.

Example 3.26. Proposition 4.1 implies that the functor

$$X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \qquad R \mapsto (\mathrm{CAlg}_R^{\mathrm{cn}})^{\Delta^0}$$

is cohesive. Moreover, it follows from [Lur17, Proposition 7.4.2.5] that for every $S \in X(R)$ there exists an S-module $L_{S/R}$ called the *relative cotangent complex*, together with, for every connective S-module M a natural equivalence

$$\mathrm{Map}_S(L_{S/R}, M[1] \otimes_R S) \xrightarrow{\sim} \mathrm{fib}_S(X(R \oplus M) \to X(R)).$$

Thus, the problem of lifting S to a $R \oplus M$-algebra $\widetilde{R}$ is equivalent to finding a map of R-algebras fitting into the diagram

$$\begin{array}{ccc} & & S \oplus (S \otimes_R M[1]) \\ & \nearrow & \downarrow \\ S & \xrightarrow{\mathrm{id}} & S. \end{array}$$

Note that, since the assignment

$$M \mapsto \mathrm{Map}_S(L_{S/R}, M[1] \otimes_R S)$$

does, in general, *not* commute with limits in M, the functor X *does not* admit a cotangent complex at S in the sense of Definition 3.23, even though the deformation theory of X at S is controlled by the module $L_{S/R}$.

Both these examples show that the module $L_{S/R}$ controls a great deal of deformation theoretic questions around the ring S. The first is largely trivial, but the fact that the problem of lifting objects between categories of algebras can be reduced to the problem of lifting maps of algebras is surprising. We now investigate how well this story can be dualized to coalgebras.

4. Deformation theory of coalgebras

We now apply the machinery reviewed in the previous section to study various deformation theoretic questions about coalgebras in the category of spectra. To this end, we first prove that the functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Cat}_\infty$ which takes a connective $\mathbb{E}_\infty$-ring spectrum R to the category $\mathrm{cCAlg}_R^{\mathrm{cn}}$ commutes with small pullbacks and limits of Postnikov towers. We then show that our deformation problems against square zero extensions with fiber M are equivalent to lifting problems against a certain map of coalgebras $\Omega_A^\infty(M) \to A$. We define formally étale coalgebras as those for which this map is an equivalence and show that, if A is dualizable and $A^\vee$ is formally étale, then A is formally étale as well. We then discuss the main deformation problem of constructing lifts along the base change $\mathrm{cCAlg}_{R^\eta}^{\mathrm{cn}} \to \mathrm{cCAlg}_R^{\mathrm{cn}}$ for some square zero extension $R^\eta \to R$ with fiber M. We show that if $A \in \mathrm{cCAlg}_R^{\mathrm{cn}}$ is formally étale then this problem admits a contractible space of solutions and that these are again formally étale. Moreover, we show that the assignment of A to its essentially unique lift $A^\eta \in \mathrm{cCAlg}_{R^\eta}^{\mathrm{cn}}$ refines to a fully faithful functor $W_p : \mathrm{cCAlg}_R^{\mathrm{cn,f\acute{e}t}} \to \mathrm{cCAlg}_{R^\eta}^{\mathrm{cn}}$.

4.1. **Descent and completion.**

Our entire discussion rests on the following descent result for modules.

Theorem 4.1 (Lurie). *Suppose we have a pullback of connective $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} R' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

such that one of the maps $\pi_0 R \to \pi_0 S$, $\pi_0 S' \to \pi_0 S$ is surjective. Then the natural map

$$\mathrm{Mod}_{R'}^{\mathrm{cn}} \to \mathrm{Mod}_R^{\mathrm{cn}} \times_{\mathrm{Mod}_S^{\mathrm{cn}}} \mathrm{Mod}_{S'}^{\mathrm{cn}}$$

is an equivalences of categories with inverse taking a point in the pullback (M, N, h), consisting of $M \in \mathrm{Mod}_R^{\mathrm{cn}}, N \in \mathrm{Mod}_{S'}^{\mathrm{cn}}$ and a homotopy $h : M \otimes_R S \simeq N \otimes_{S'} S$, to $M \times_{M \otimes_R S} N$ with the induced R'-module structure.

Proof. [Lur21, Theorem 16.2.0.2.] □

Corollary 4.2. *In the setting of Theorem 4.1, the map*

$$\mathrm{Mod}^{\mathrm{cn}}_{R'} \to \mathrm{Mod}^{\mathrm{cn}}_{R} \times_{\mathrm{Mod}^{\mathrm{cn}}_{S}} \mathrm{Mod}^{\mathrm{cn}}_{S'}$$

induces equivalences

$$\mathrm{cCAlg}^{\mathrm{cn}}_{R'} \xrightarrow{\sim} \mathrm{cCAlg}^{\mathrm{cn}}_{R} \times_{\mathrm{cCAlg}^{\mathrm{cn}}_{S}} \mathrm{cCAlg}^{\mathrm{cn}}_{S'},$$
$$\mathrm{CAlg}^{\mathrm{cn}}_{R'} \xrightarrow{\sim} \mathrm{CAlg}^{\mathrm{cn}}_{R} \times_{\mathrm{CAlg}^{\mathrm{cn}}_{S}} \mathrm{CAlg}^{\mathrm{cn}}_{S'}.$$

Proof. Recall that by Proposition 2.2 the forgetful functor

$$\mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$$

commutes with limits. In particular, a pullback of presentably monoidal categories and strong monoidal functors

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$

inherits a natural symmetric monoidal structure. Moreover, by Lemma 2.6 we have an equivalence

$$\mathrm{cCAlg}(\mathcal{C} \times_{\mathcal{D}} \mathcal{E}) \simeq \mathrm{cCAlg}(\mathcal{C}) \times_{\mathrm{cCAlg}(\mathcal{D})} \mathrm{cCAlg}(\mathcal{E}),$$

so the first claim follows. The argument for $\mathrm{CAlg}^{\mathrm{cn}}$ is exactly the same. □

This tells us that for a connective R-coalgebra A, a connective S'-coalgebra B and an equivalence $A \otimes_R S \simeq B \otimes_{S'} S$ the pullback of spectra $A \times_{A \otimes_R S} B$ inherits a natural R'-coalgebra structure and in fact every connective R'-coalgebra is of this form. We can immediately apply this to the deformation theory of coalgebras. Let $R^\eta \to R$ be a square zero extension classified by a derivation $\eta : R \to M[1]$ and let $A \in \mathrm{cCAlg}^{\mathrm{cn}}_R$ be a coalgebra. Then by Theorem 4.1, the pullback of ring spectra

$$\begin{array}{ccc} R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ R & \xrightarrow[(\mathrm{id},\eta)]{} & R \oplus M[1] \end{array}$$

induces a pullback of ∞-categories

$$\begin{array}{ccc} \mathrm{cCAlg}^{\mathrm{cn}}_{R^\eta} & \longrightarrow & \mathrm{cCAlg}^{\mathrm{cn}}_{R} \\ \downarrow & & \downarrow \\ \mathrm{cCAlg}^{\mathrm{cn}}_{R} & \longrightarrow & \mathrm{cCAlg}^{\mathrm{cn}}_{R \oplus M[1]} \end{array} .$$

Thus, a R^η-coalgebra is given by a R-coalgebra A together with an equivalence of the two base changes $A \otimes_0 (R \oplus M[1]) \simeq A \otimes_\eta (R \oplus M[1])$. If such an equivalence exists, the R^η-coalgebra is classified by a map $d : A \to A \otimes M[1]$ in the sense that its underlying spectrum can be computed as the pullback

$$\begin{array}{ccc} A^d & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ A & \xrightarrow[(\mathrm{id},d)]{} & A \oplus A \otimes M[1]. \end{array}$$

However, the module $A \oplus A \otimes M[1]$ does not admit a natural coalgebra structure, exhibiting the map $A \xrightarrow{(\mathrm{id},0)} A \oplus A \otimes M[1]$ as a coalgebra morphism. Even if this was the case, this would not recover the coalgebra structure on A^d since limits of coalgebras are not formed underlying. This is one of the main defects of the deformation theory of coalgebras compared the one of algebras. For $S \in \mathrm{CAlg}_R^{\mathrm{cn}}$, we can forget the $R \oplus M[1]$-algebra structure on $S \otimes_R M[1]$ and exhibit the map $S \xrightarrow{(\mathrm{id},0)} S \oplus S \otimes_R M[1]$ as a morphism of underlying R-algebras. This fact is exploited by Lurie in [Lur17, Proposition 7.4.2.5] to give a full classification of connective $\mathbb{E}_\infty$-algebras over a square zero extension in terms of algebras over the ground ring. However for coalgebras, there is no forgetful functor which gives us the "underlying" R-coalgebra, only the mysterious right adjoint $\mathrm{cCAlg}_{R\oplus M[1]} \to \mathrm{cCAlg}_R$. The goal of this section is to understand how big of a problem this poses and which parts of the story carry over regardless.

Lemma 4.3. *Let $\cdots \to E_2 \to E_1 \to E_0$ be a diagram of spectra. Suppose we are given $L \geq 0$ such that for all $\ell' \geq \ell \geq L$ the map $E_{\ell'} \to E_\ell$ is m-connective. Then for any $\ell \geq L$ the map*

$$\varprojlim_n E_n \to E_\ell$$

is $m-1$-connective.

Proof. Writing $F_{\ell',\ell} = \mathrm{fib}(E_{\ell'} \to E_\ell)$ and $F_\ell = \mathrm{fib}(\lim_n E_n \to E_\ell)$ we want to show that F_ℓ is m-connective. Indeed, since limits are exact, we have that

$$F_\ell \simeq \lim_{\ell'>\ell} F_{\ell',\ell} \simeq \mathrm{fib}\left(\prod_{\ell'>\ell} F_{\ell',\ell} \to \prod_{\ell'>\ell} F_{\ell'-1,\ell}\right).$$

Thus, since $\mathrm{Sp}_{\geq m}$ is closed under products and the fiber of a map of m-connective spectra is $m-1$-connective, the claim follows. □

Proposition 4.4. *Let R be a connective $\mathbb{E}_\infty$-ring. Then the natural map*

$$\mathrm{Mod}_R^{\mathrm{cn}} \to \varprojlim_n \mathrm{Mod}_{\tau_{\leq n}R}^{\mathrm{cn}} \quad M \mapsto M \otimes_R \tau_{\leq n}R$$

is an equivalence of categories.

Proof. We write $R_n := \tau_{\leq n}R$. The functor admits a right adjoint which takes $(M_n) \in \varprojlim_n \mathrm{Mod}_{R_n}^{\mathrm{cn}}$ to the limit $\lim_n M_n$ which inherits a natural action by $\lim_n R_n \simeq R$. Now let $N \in \mathrm{Mod}_R$. Since taking limits is exact, the counit of the adjunction sits in a fiber sequence

$$\lim_n \operatorname{fib}(N \to N \otimes_R R_n) \to N \xrightarrow{\eta} \lim_n (N \otimes_R R_n).$$

where we compute for the left hand term that

$$\operatorname{fib}(N \to N \otimes_R R_n) \simeq \operatorname{fib}(N \otimes_R R \to N \otimes_R R_n) \simeq N \otimes_R \operatorname{fib}(R \to R_n).$$

Now, since $R_n = \tau_{\leq n} R$, the connectivity of $\operatorname{fib}(R \to R_n)$ increases with n. Hence, since R and N are connective, so does the connectivity of the tensor product $N \otimes_R \operatorname{fib}(R \to R_n)$ which implies that $\varprojlim_n N \otimes_R \operatorname{fib}(R \to R_n) \simeq 0$. Thus, the counit $N \to \lim_n (N \otimes_R R_n)$ is an equivalence.

Now let $(M_n) \in \varprojlim_n \operatorname{Mod}^{\mathrm{cn}}_{R_n}$ and write $M = \lim_n M_n$. We need to show that the natural map

$$\varepsilon_k : M \otimes_R R_k \to R_k$$

is an equivalence for each k. We do this by showing that it is m-connective for any $m \geq 0$. Indeed, for any such m there exists an integer L such that for all $\ell \geq \ell' > L$ the natural map $R_\ell \to R_{\ell'}$ is m-connective. Since $M_{\ell'} \simeq M_\ell \otimes_{R_\ell} R_{\ell'}$ we have a fiber sequence

$$M_\ell \otimes_{R_\ell} (\operatorname{fib}(R_\ell \to R_{\ell'})) \to M_\ell \to M_{\ell'}.$$

Hence, since $\operatorname{fib}(R_\ell \to R_{\ell'})$ is m-connective and R_ℓ and M_ℓ are connective, the tensor product $M_\ell \otimes_{R_\ell} \operatorname{fib}(R_\ell \to R_{\ell'})$ is m-connective as well. Thus, for fixed m and k we may apply Lemma 4.3 to obtain $\ell > k$ such that the maps $M \to M_\ell$ and $R \to R_\ell$ are both m-connective. Finally, the map

$$\varepsilon_k : M \otimes_R R_k \to M_\ell \otimes_{R_\ell} R_k \simeq M_k$$

is given by the colimit of the induced map between the bar resolutions

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
M \otimes R \otimes R_k & \longrightarrow & M_\ell \otimes R_\ell \otimes R_k \\
\downarrow\downarrow & & \downarrow\downarrow \\
M \otimes R_k & \longrightarrow & M_\ell \otimes R_k
\end{array} .$$

Denote by F_n the fiber of the map $M \otimes R^{\otimes n} \otimes R_k \to M_\ell \otimes R_\ell^{\otimes n} \otimes R_k$. Since the tensor product of m-connective maps is again m-connective, the fiber F_n is m-connective. Thus, by exactness of colimits, we obtain a fiber sequence

$$\operatorname{colim} F_n \to M \otimes_R R_k \xrightarrow{\varepsilon_k} M_k$$

and finally, since taking colimits preserves connectivity, this shows that the map ε_k is m-connective. Since m was arbitrary, the map ε_k is in fact an equivalence which completes the proof. □

Corollary 4.5. *Let R be a connective $\mathbb{E}_\infty$-ring, then the natural maps*

$$\mathrm{cCAlg}_R^{\mathrm{cn}} \to \varprojlim_n \mathrm{cCAlg}^{\mathrm{cn}}_{\tau_{\leq n}R}$$

$$\mathrm{CAlg}_R^{\mathrm{cn}} \to \varprojlim_n \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\leq n}R}$$

are equivalences.

Proof. Analogously to Proposition 4.1, this follows by observing that the map

$$\mathrm{Mod}_R^{\mathrm{cn}} \to \varprojlim_n \mathrm{Mod}^{\mathrm{cn}}_{\tau_{\leq n}R} \quad M \mapsto M \otimes_R \tau_{\leq n}R$$

from Proposition 4.6 is strong monoidal. □

Corollary 4.6. *For any $n \in \mathbb{N}$ the functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which takes a connective $\mathbb{E}_\infty$-ring R to the space $(\mathrm{cCAlg}_R^{\mathrm{cn}})^{\Delta^n} := \mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^n, \mathrm{cCAlg}_R^{\mathrm{cn}})$ is cohesive and nilcomplete.*

Proof. This is clear since the functors $\mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^n, -) : \mathrm{Cat}_\infty \to \mathcal{S}$ commute with limits. □

In particular, we immediately obtain the existence of a tangent complex which controls lifts of coalgebras.

Corollary 4.7. *Let $A \in X(R) = (\mathrm{cCAlg}_R^{\mathrm{cn}})^{\Delta^0}$ and let $R^\eta \to R$ be a square zero extension classified by a derivation $R \xrightarrow{\eta} M[1]$. Then the sequence $\{X_A^{R\oplus M[n]}\}_{n\in\mathbb{N}}$ defines a spectrum $T^M_{X_A}$, yielding an obstruction class $A^\eta \in \pi_{-1}T^M_{X_A}$ such that, the space of deformations of A to a R^η-coalgebra is non-empty if and only if A^η vanishes and, in this case, is a torsor under $\Omega^\infty T^M_{X_A} = X_A^{R\oplus M}$.*

Proof. This is exactly Proposition 3.19. □

4.2. **Formally étale coalgebras and derivations.** We now want to introduce a class of coalgebras for which all these lifting problems have a contractible space of solutions and find an alternate description of the Tangent Complex via a notion of derivation. We begin with the following general observation.

Construction 4.8. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $f^* : \mathcal{C} \to \mathcal{D}$ with right adjoint $f_! : \mathcal{D} \to \mathcal{C}$. Moreover, suppose we have $g^* : \mathcal{D} \to \mathcal{C}$ such that $g^*f^* \simeq \mathrm{id}_{\mathcal{C}}$ and consider the natural transformation

$$\pi : f_!f^* \xrightarrow{\sim} g^*f^*f_!f^* \to g^*f^* \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$$

defined as the whiskering of the counit $\varepsilon : f^*f_! \to \mathrm{id}$ as in the diagram

$$\mathcal{C} \xrightarrow{f^*} \mathcal{D} \underset{\mathrm{id}}{\overset{f^*f_!}{\Downarrow}} \mathcal{D} \xrightarrow{g^*} \mathcal{C}.$$

Unraveling the definition we see that for each $B, A \in \mathcal{C}$ the composition

$$\mathrm{Map}_{\mathcal{C}}(B, f_!f^*A) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(f^*B, f^*A) \xrightarrow{g^*} \mathrm{Map}_{\mathcal{C}}(B, A)$$

takes $\psi : B \to f_! f^* A$ to the composite $\pi_A \circ \psi$. Thus, for each $\varphi : B \to A$ we have an equivalence between the fiber

$$\mathrm{fib}_\varphi \left(\mathrm{Map}_{\mathcal{D}}(f^* B, f^* A) \xrightarrow{g^*} \mathrm{Map}_{\mathcal{C}}(B, A) \right)$$

and the mapping space

$$\mathrm{Map}_{\mathcal{C}_{/A}}((B \xrightarrow{\varphi} A), (f_! f^* A \xrightarrow{\eta_A} A)).$$

Example 4.9. Let $R \in \mathrm{CAlg}$ and let $X(-) = \mathrm{cCAlg}_{-}$. Moreover, let $A, B \in \mathrm{cCAlg}_R$, $M \in \mathrm{Mod}_R$ and suppose we are given a map of coalgebras $\varphi : B \to A$. Denote the natural inclusion as $f_M : R \to R'$ together with it's section $g_M : R \oplus M \to R$. Recall the adjunction

$$\mathrm{cCAlg}_R \underset{f_{M,!}}{\overset{f_M^*}{\underset{\perp}{\rightleftarrows}}} \mathrm{cCAlg}_{R \oplus M}$$

described in Remark 2.8 and set

$$\Omega_A^\infty(M) := f_{M,!} f_M^* A \in \mathrm{cCAlg}_R.$$

We want to analyze the space of lifts of ϕ to a map $f_M^* B \to f_M^* A$. Since $g_M f_M = \mathrm{id}$, we also have $g_M^* f_M^* = \mathrm{id}$. Thus, Construction 4.8 gives a natural map of R-coalgebras

$$\pi_A : \Omega_A(M) \to A$$

exhibiting the fiber

$$\mathrm{fib}_\varphi \left(\mathrm{Map}_{\mathrm{cCAlg}_{R \oplus M}}(f_M^* B, f_M^* A) \to \mathrm{Map}_{\mathrm{cCAlg}_R}(B, A) \right)$$

as the space of lifts in the diagram

$$\begin{array}{ccc} & & \Omega_A^\infty(M) \\ & \nearrow & \downarrow \pi_A \\ B & \longrightarrow & A. \end{array}$$

Moreover, if we assume that A is connective, the problem of constructing a lift of the coalgebra A itself can now be described as follows: Let $X = (\mathrm{cCAlg}_{-}^{\mathrm{cn}})^{\Delta^1}$ and $A \in X(R)$. Then since X is cohesive by Corollary 4.6 we have an equivalence

$$X_A^{R \oplus M} \simeq \Omega X_A^{R \oplus M[1]} = \mathrm{fib}_{\mathrm{id}_A}(\mathrm{Map}_{\mathrm{cCAlg}_{\mathrm{R \oplus M[1]}}}(f_M^* A, f_M^* A) \to \mathrm{Map}_{\mathrm{cCAlg}}(A, A)).$$

Hence, $X_A^{R \oplus M}$ is given by the space of lifts in the diagram

$$\begin{array}{ccc} & & \Omega_A(M[1]) \\ & \nearrow & \downarrow \pi_A \\ A & \xrightarrow{\mathrm{id}} & A. \end{array}$$

Suppose we have an R-coalgebra A for which the lifting problems above admit a contractible space of solutions. Then, for every $B \in \mathrm{cCAlg}_R$ composing with π_A gives an equivalence

$$\mathrm{Map}_{\mathrm{cCAlg}_R}(B, \Omega^\infty_A(M)) \xrightarrow{\sim} \mathrm{Map}(B, A).$$

Hence, by the Yoneda Lemma the map $\Omega^\infty_A(M) \xrightarrow{\pi_A} A$ is an equivalence for all $M \in \mathrm{Mod}_R$. Moreover, since the composition

$$A \xrightarrow{\varepsilon} \Omega^\infty_A M \xrightarrow{\pi_A} A$$

is homotopic to the identity, the counit $A \xrightarrow{\varepsilon} \Omega^\infty_A M$ is an equivalence if and only if π_A is. This motivates the following definition:

Definition 4.10. Let R be a connective $\mathbb{E}_\infty$-ring, then for each R-coalgebra A we have a functor

$$\mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Mod}_R \qquad M \mapsto C_A(M) := \mathrm{cofib}(A \xrightarrow{\varepsilon} \Omega^\infty_A(M)).$$

We say that $A \in \mathrm{cCAlg}_R$ is *formally étale* if $C_M(A) \simeq 0$ for all M and denote the full subcategory spanned by the formally étale coalgebras by $\mathrm{cCAlg}_R^{\mathrm{fét}} \subseteq \mathrm{cCAlg}_R$. Moreover, given a map $B \xrightarrow{\varphi} A$, we define the *space of derivations* $B \to C_A(M)$ as the mapping space

$$\mathrm{Der}_\varphi(B, C_A(M)) := \mathrm{Map}_{\mathrm{cCAlg}_{R/A}}(B, \Omega^\infty_A M).$$

Warning 4.11. The notation $\Omega^\infty_A M$ is at this point merely suggestive of the dual story in the algebra world. There we had seen that, given a map $\varphi : R \to S$ we have equivalences

$$\mathrm{Map}_{\mathrm{CAlg}_{/S}}(R, S \oplus M) \simeq \mathrm{Map}_{\mathrm{CAlg}_{/R}}(R, R \oplus \varphi_* M) \simeq \mathrm{Map}_{\mathrm{CAlg}_{/R}}(R, \Omega^\infty(\varphi_* M))$$

Where Ω^∞ denotes the infinite loop space map

$$\mathrm{Mod}_R \simeq \mathrm{Sp}(\mathrm{CAlg}_{/R}) \xrightarrow{\Omega^\infty} \mathrm{CAlg}_{/R}$$

from Proposition 3.1. However it is at present unclear whether the functor

$$\mathrm{Mod}_R \to \mathrm{cCAlg}_{/A} \qquad M \mapsto \Omega^\infty_A M$$

admits a similar description. This is the main obstacle to overcome in trying to deduce the existence of a cotangent complex for a coalgebra A.

A coalgebra $A \in \mathrm{cCAlg}_R$ is formally étale if and only $C_A(M)$ admits only trivial derivations, which is the case if and only if the map $\Omega^\infty_A M \xrightarrow{\pi_A} A$ is an equivalence for all $M \in \mathrm{Mod}_R$. Thus, we can think of derivations into $C_A(M)$ as a measuring how far A is from being formally étale. To see that this property is reasonable, i.e. one that is satisfied by a nontrivial class of coalgebras, we now consider the case where A is dualizable.

Proposition 4.12. *Let R be an $\mathbb{E}_\infty$-ring, $A, B \in \mathrm{cCAlg}_R$ with A dualizable and $M \in \mathrm{Mod}_R$. Write $R' = R \oplus M$ and $A_{R'}$ respectively $B_{R'}$ for the basechange along the inclusion $R \to R'$. Then for every $\varphi : B \to A$ there is a natural equivalence*

$$\mathrm{Der}_\phi(B, C_A(M)) \simeq \mathrm{Map}_{A^\vee}(L_{A^\vee/R}, \varphi^\vee_* \mathrm{map}_R(B, M)).$$

Proof. We know from Example 4.9 that the space of derivations $B \to C_A(M)$ is equivalent to the fiber

$$F_\varphi := \mathrm{fib}_\varphi\left(\mathrm{Map}_{\mathrm{cCAlg}_{R'}}(B_{R'}, A_{R'}) \to \mathrm{Map}_{\mathrm{cCAlg}_R}(B, A)\right)$$

Since A is dualizable, so is $A_{R'}$ with dual given by $A_{R'}^\vee \simeq A^\vee \otimes_R R'$. Thus, applying $(-)^\vee$ we get an equivalence

$$\begin{aligned}\mathrm{Map}_{\mathrm{cCAlg}_{R'}}(B_{R'}, A_{R'}) &\simeq \mathrm{Map}_{\mathrm{CAlg}_{R'}}(A^\vee \otimes_R R', \mathrm{map}_{R'}(B_{R'}, R')) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_R}(A^\vee, \mathrm{map}_R(B, R'))\end{aligned}$$

and similarly

$$\mathrm{Map}_{\mathrm{cCAlg}_R}(B, A) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(A^\vee, B^\vee).$$

Thus, F_φ is given by

$$\begin{aligned}F_\varphi &\simeq \mathrm{fib}_{\varphi^\vee}\left(\mathrm{Map}_{\mathrm{CAlg}_R}(A^\vee, \mathrm{map}_R(B, R')) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A^\vee, B^\vee)\right) \\ &\simeq \mathrm{Map}_{(\mathrm{CAlg}_R)_{/B^\vee}}(A^\vee, \mathrm{map}_R(B, R')),\end{aligned}$$

i.e. the space of lifts in the diagram

$$\begin{array}{ccc} & & \mathrm{map}_R(B_R, R') \\ & \nearrow & \downarrow \\ A^\vee & \longrightarrow & B^\vee. \end{array}$$

Since $R' \to R$ is a split square zero extension with fiber M, the map $\mathrm{map}_R(B, R') \to B^\vee$ is a square zero extension as well with fiber given by $\mathrm{map}_R(B, M)$. Hence, we have that

$$F_\varphi \simeq \mathrm{Map}_{A^\vee}(L_{A_R^\vee/R}, \varphi_*^\vee \mathrm{map}_R(B, M))$$

as claimed. □

Corollary 4.13. *Let R be an $\mathbb{E}_\infty$-ring and $A \in \mathrm{CAlg}_R$ be dualizable such that $L_{A^\vee/R} \simeq 0$. Then A is formally étale.*

Remark 4.14. It is unclear whether the converse of Corollary 4.13 holds. From Proposition 4.12 we can only deduce that

$$\mathrm{Map}_{A^\vee}(L_{A^\vee/R}, \varphi_* \mathrm{map}_R(B, M)) \simeq 0$$

for each coalgebra B, R-module M and morphism of algebras $\varphi : A^\vee \to B^\vee$.

Construction 4.15. Let R be a connective $\mathbb{E}_\infty$-ring and $B \xrightarrow{\varphi} A$ a morphism of connective R-coalgebras. Then the functor

$$\mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S} \quad M \mapsto \mathrm{Der}_\varphi(B, C_A(M))$$

is reduced and excisive. Indeed, denoting $X(R) = \mathrm{cCAlg}_R^{\mathrm{cn}}$, we may write

$$\mathrm{Der}_\varphi(B, C_A(M)) \simeq \mathrm{fib}_{(A_{R\oplus M}, B_{R\oplus M})}\left((X^{\Delta^1})_\varphi^{R\oplus M} \xrightarrow{\mathrm{ev}_0, \mathrm{ev}_1} (X^{\Delta^0})_A^{R\oplus M} \times (X^{\Delta^0})_B^{R\oplus M}\right),$$

so both properties follow from Proposition 3.14 since X^{Δ^n} is cohesive for every n. Thus, as in Construction 3.15, we can associate to the functor $\mathrm{Der}_\varphi(B, C_A(M))$ a spectrum which we denote $\mathrm{der}_\varphi(B, C_A(M))$.

Lemma 4.16. *Let $X(R) = (\mathrm{cCAlg}_R^{\mathrm{cn}})^{\Delta^0}$ and M be a connective R-module. Then we have a natural equivalence $T_{X_A}^M \simeq \mathrm{der}(A, C_A(M[1]))$.*

Proof. The space $\mathrm{X}_A^{R\oplus M[1]}$ is pointed by the coalgebra $A' := A \otimes_R (R \oplus M[1])$, and since X is cohesive, we have that $\mathrm{X}_A^{R\oplus M} \simeq \Omega_{A'} X_A^{R\oplus M[1]}$. This loop space is then given by

$$\begin{aligned}\Omega_{A'} X_A^{R\oplus M[1]} &\simeq \mathrm{fib}_{\mathrm{id}_A}\Big(\mathrm{Map}_{\mathrm{cCAlg}_{R\oplus M[1]}}(A', A') \to \mathrm{Map}_{\mathrm{cCAlg}_R}(A, A)\Big)\\ &\simeq \mathrm{Der}_{\mathrm{id}}(A, C_A(M[1])).\end{aligned}$$

This equivalence is natural in M and thus induces an equivalence of the associated spectra $T_{X_A}^{R\oplus M} \simeq \mathrm{der}(A, C_A(M[1]))$, as claimed. □

Corollary 4.17. *Let $A \in \mathrm{cCAlg}_R^{\mathrm{cn}}$ be formally étale and $R^\eta \to R$ a square zero extension. Then the fiber*

$$\mathrm{fib}_A(\mathrm{cCAlg}_{R^\eta} \to \mathrm{cCAlg}_R)$$

is contractible, i.e. A admits an essentially unique lift to a R^η-coalgebra.

We now discuss how to lift maps of coalgebras, with the goal of making lifts of formally étale coalgebras functorial. To this end, we first compute the fibers of $\mathrm{cCAlg}_{(-)}^{\Delta^1}$ in terms of the spaces of derivations introduced above.

Proposition 4.18. *Let $X(-) = (\mathrm{cCAlg}_-^{\mathrm{cn}})^{\Delta^1}$, R an $\mathbb{E}_\infty$-ring and $(A \xrightarrow{\varphi} B) \in X(R)$. Then for every connective R-module M the fiber $X_\varphi^{R\oplus M}$ can be computed as the pullback*

$$X_\varphi^{R\oplus M} \simeq \mathrm{Der}_{\mathrm{id}}(B, C_B(M[1])) \times_{\mathrm{Der}_\varphi(B, C_A(M[1]))} \mathrm{Der}_{\mathrm{id}}(A, C_A(M[1])).$$

Proof. Since X^{Δ^1} is cohesive, the space $(X^{\Delta^1})_\varphi^{R\oplus M}$ fits into a pullback diagram

$$\begin{array}{ccc} X_\varphi^{R\oplus M} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X_\varphi^{R\oplus M[1]} \end{array}$$

where the maps $* \to X_\varphi^{R\oplus M[1]}$ are given by the base changed morphisms $\varphi \otimes_R (R \oplus M[1])$. If $\mathcal{C}$ is any category and $(\psi : A \to B) \in \mathcal{C}^{\Delta^1}$, then $\Omega_\psi \mathcal{C}^{\Delta^1}$ is given by the pullback

$$\begin{array}{ccc} \Omega_\psi \mathcal{C}^{\Delta^1} & \longrightarrow & \mathrm{Aut}_{\mathcal{C}}(A) \\ \downarrow & & \downarrow {\scriptstyle \psi\circ -} \\ \mathrm{Aut}_{\mathcal{C}}(B) & \xrightarrow[-\circ\psi]{} & \mathrm{Map}_{\mathcal{C}}(A, B) \end{array} .$$

Thus, we can compute the loop space $\Omega X_\varphi^{R\oplus M[1]}$ as the pullback

$$\begin{array}{ccc} \Omega X_\varphi^{R\oplus M[1]} & \longrightarrow & \mathrm{Der}_{\mathrm{id}}(A, C_A(M[1])) \\ \downarrow & & \downarrow \\ \mathrm{Der}_{\mathrm{id}}(B, C_B(M[1])) & \longrightarrow & \mathrm{Der}_{\mathrm{id}}(B, C_A(M[1])) \end{array}$$

as claimed. ☐

Hence, we can think of a lift $\psi \in X_\varphi^{R\oplus M}$ as being given by two derivations $\mu : B \to \Omega_B^\infty M[1]$ and $\nu : A \to \Omega_A^\infty M[1]$ together with a homotopy

$$\begin{array}{ccc} B & \xrightarrow{\mu} & \Omega_B^\infty M[1] \\ \varphi\uparrow & \nearrow\!\!\!\nearrow & \uparrow \\ A & \xrightarrow[\nu]{} & \Omega_A^\infty M[1] \end{array}$$

lying over the diagram in $\mathrm{cCAlg_R}$

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ \varphi\uparrow & \nearrow\!\!\!\nearrow_{\mathrm{id}} & \uparrow\varphi \\ B & \xrightarrow[\mathrm{id}]{} & B \end{array} .$$

In particular, if A is formally étale this means that the space of lifts of φ is equivalent to the space of lifts of B to an $R \oplus M$-coalgebra. In fact, this property characterizes formally étale coalgebras, as we will see in the following proposition.

Proposition 4.19. *Let R be a connective $\mathbb{E}_\infty$-ring and $A \in \mathrm{cCAlg}_R^{\mathrm{cn}}$ and write $\mathcal{X}(R) = \mathrm{cCAlg}_R^{\mathrm{cn}}$. Then A is formally étale if and only if for every $B \in \mathrm{cCAlg}_R^{\mathrm{cn}}, M \in \mathrm{Mod}^{cn}$ and every morphism $\varphi : B \to A$, the map $T^M_{(\mathcal{X}^{\Delta^1})_\varphi} \to T^M_{(\mathcal{X}^{\Delta^0})_B}$ induced by the evaluation $\mathcal{X}^{\Delta^1} \xrightarrow{\mathrm{ev}_0} \mathcal{X}^{\Delta^0}$ is an equivalence.*

Proof. Denote the fiber of the map

$$(\mathcal{X}^{\Delta^1})_\varphi^{R\oplus M} \to (\mathcal{X}^{\Delta^0})_B^{R\oplus M} \simeq \mathrm{Der}_{\mathrm{id}}(B, C_B M)$$

over some point $B' \in (\mathcal{X}^{\Delta^0})_B^{R\oplus M}$ by $F_{B'}^M$. Then by definition of the tangent complex the map $T^M_{(\mathcal{X}^{\Delta^1})_\varphi} \to T^M_{(\mathcal{X}^{\Delta^0})_B}$ is an equivalence if and only if we have $F_{B'}^m \simeq 0$ for all $M \in \mathrm{Mod}_R^{\mathrm{cn}}, B' \in (\mathcal{X}^{\Delta^0})_B^{R\oplus M}$. The two pasted pullback squares

$$\begin{array}{ccccc} F_{B'}^M & \longrightarrow & (\mathcal{X}^{\Delta^1})_\varphi^{R\oplus M} & \longrightarrow & \mathrm{Der}_{\mathrm{id}}(A, C_A(M[1])) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathrm{Der}_{\mathrm{id}}(B, C_B(M[1])) & \longrightarrow & \mathrm{Der}_\varphi(B, C_A(M[1])) \end{array}$$

yield a fiber sequence

$$F_M \to \mathrm{Der}_{\mathrm{id}}(A, C_A(M[1])) \xrightarrow{-\circ\varphi} \mathrm{Der}_\varphi(B, C_A(M[1])).$$

Hence, the "only if" direction holds. Moreover, we see that if $F^M_{B'} \simeq 0$ for every $M \in \mathrm{Mod}^{\mathrm{cn}}_R$, $B' \in (\mathcal{X}^{\Delta^0})^{R\oplus M}_B$ and any morphism $B \xrightarrow{\varphi} A$, we obtain a zigzag of equivalences

$$\mathrm{Der}_\varphi(B, C_A(M[1])) \xleftarrow{\sim} \mathrm{Der}_{\mathrm{id}}(A, C_A(M[1])) \xrightarrow{\sim} \mathrm{Der}_0(0, C_A(M[1])) \simeq *,$$

where $0 \in \mathrm{cCAlg}_k$ denotes the initial coalgebra. Thus, we have that

$$\mathrm{Der}_\varphi(B, C_A(M)) \simeq \Omega\mathrm{Der}_\varphi(B, C_A(M[1])) \simeq *$$

as claimed. □

Corollary 4.20. *Let R be a connective ring spectrum and $A \in \mathrm{cCAlg}^{\mathrm{cn}}_R$ be formally étale. For a square zero extension $R^\eta \to R$ with fiber M denote by A^η the essentially unique lift of A to a connective R^η-coalgebra. Then A^η is also formally étale.*

Proof. Let $B \to A^\eta$ be any map of R^η coalgebras and write $\mathcal{X}(-) = \mathrm{cCAlg}^{\mathrm{cn}}_{-}$. Then for any $N \in \mathrm{Mod}^{\mathrm{cn}}_{R^\eta}$ we need to show that the induced map

$$T^M_{\mathcal{X}^{\Delta^1}_\varphi} \xrightarrow{\sim} T^M_{\mathcal{X}^{\Delta^0}_B}$$

is an equivalence. Arguing as in the proof of Proposition 3.21, we see that N sits in a cofiber sequence

$$M \otimes_R (R \otimes_{R^\eta} N) \to N \to R \otimes_{R^\eta} N,$$

where the R^η-action on the outer terms factors through R. Thus, writing $B' \simeq B \otimes_{R^\eta} R$ and $\varphi' = \varphi_{R^\eta} : B' \to A$ and using that the tangent complex functors are excisive, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
T^{M\otimes_R(R\otimes_{R^\eta}N)}_{\mathcal{X}^{\Delta^1}_{\varphi'}} & \xleftarrow{\sim} & T^{M\otimes_R(R\otimes_{R^\eta}N)}_{\mathcal{X}^{\Delta^1}_{\varphi}} & \longrightarrow & T^N_{\mathcal{X}^{\Delta^1}_\varphi} & \longrightarrow & T^{R\otimes_{R^\eta}N}_{\mathcal{X}^{\Delta^1}_\varphi} & \xrightarrow{\sim} & T^{R\otimes_{R^\eta}N}_{\mathcal{X}^{\Delta^1}_{\varphi'}} \\
\downarrow{\sim} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\sim} \\
T^{M\otimes_R(R\otimes_{R^\eta}N)}_{\mathcal{X}^{\Delta^0}_{B'}} & \xleftarrow{\sim} & T^{M\otimes_R(R\otimes_{R^\eta}N)}_{\mathcal{X}^{\Delta^0}_{B}} & \longrightarrow & T^N_{\mathcal{X}^{\Delta^0}_B} & \longrightarrow & T^{R\otimes_{R^\eta}N}_{\mathcal{X}^{\Delta^0}_B} & \xrightarrow{\sim} & T^{R\otimes_{R^\eta}N}_{\mathcal{X}^{\Delta^0}_{B'}}
\end{array}$$

where the inner two horizontal maps in each row form a cofiber sequence. The outer horizontal maps are the base change equivalences from Proposition 3.20 and the outer vertical maps are equivalences since by assumption $A = A^\eta \otimes_{R^\eta} R$ is formally étale. Thus, the middle map $T^N_{\mathcal{X}^{\Delta^1}_\varphi} \to T^N_{\mathcal{X}^{\Delta^0}_B}$ is an equivalence as well, so by Proposition 4.19 the R^η-coalgebra A^η is formally étale. □

Corollary 4.21. *Let $A, B \in \mathrm{cCAlg}^{\mathrm{cn}}_R$ with A formally étale and let $R^\eta \to R$ be a square zero extension. Suppose we are given lifts A' and B' of A and B respectively to $\mathrm{cCAlg}^{\mathrm{cn}}_{R^\eta}$. Then the natural map*

$$\mathrm{Map}_{\mathrm{cCAlg}_{R^\eta}}(B', A') \to \mathrm{Map}_{\mathrm{cCAlg}_R}(B, A)$$

is a homotopy equivalence.

Proof. This is immediate from Proposition 4.18. □

Proposition 4.22. *Let R be a connective $\mathbb{E}_\infty$-ring, $R^\eta \to R$ a square zero extension and denote by $\mathcal{C} \subseteq \mathrm{cCAlg}^{\mathrm{cn}}_{R^\eta}$ the full subcategory spanned by those coalgebras A such that $A \otimes_{R^\eta} R$ is formally étale. Then the functor*

$$\mathcal{C} \to \mathrm{cCAlg}^{\mathrm{cn,fét}}_{\mathbb{F}_p} \qquad A \mapsto A \otimes_{R^\eta} R$$

is fully faithful and essentially surjective. In particular the quasi-inverse defines a functor

$$W_\eta : \mathrm{cCAlg}^{\mathrm{cn,fét}}_{R} \to \mathrm{cCAlg}^{\mathrm{cn}}_{R^\eta}$$

which is fully faithful and satisfies $W_\eta(A) \otimes_{R^\eta} R \simeq A$. Moreover, up to contractible choice $W_\eta(A)$ is the unique connective R^η-coalgebra with this property.

Proof. Combine Corollary 4.17 and Corollary 4.21. □

This means that we can lift étale coalgebras not just uniquely, but functorially against square zero extensions. Since these lifts are again formally étale by Corollary 4.20, we see that we can iterate this process. This will be the main theme of the next section.

5. Applications to p-adic homotopy theory

For this chapter, fix a prime p. We first discuss deformations of coalgebras from $\mathbb{F}_p$ to the p-adic integers and further to the p-completed sphere $\mathbb{S}^\wedge_p$ which leads us to the question of how coalgebras behave with respect to p-completion. We introduce the notion of a p-complete coalgebra and show that this is well behaved with respect to the deformation theory discussed in the previous chapter. We then use this to iterate Proposition 4.22 and prove our main results, namely the existence of Witt Vectors and spherical Witt Vectors for formally étale coalgebras. Then we specialize to the case of homology coalgebras, show that for a finite space X the coalgebra $\mathbb{F}_p[X]$ is formally étale, and answer our initial question about the relation between $\mathbb{S}[X]^\wedge_p$ and $\mathbb{F}_p[X]$

5.1. Coalgebras and p-completion.

We have seen that the functors that interest us are all *nilcomplete*. For a nilcomplete functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and a connective $\mathbb{E}_\infty$-ring R, we can construct lifts from $X(\pi_0 R)$ to $X(R)$ inductively along the Postnikov tower

$$\cdots \to \tau_{\leq 2} R \to \tau_{\tau \leq 1} R \to \tau_{\leq 0} R = \pi_0 R.$$

This is however not quite enough to obtain our goal of lifting from $\mathbb{F}_p$ to the p-completed sphere, we first need to pass to $\mathbb{Z}_p = \pi_0 \mathbb{S}^\wedge_p$. Explicitly, this means constructing lifts against the tower

$$\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to \mathbb{F}_p$$

which is clearly presents a different problem. With the machinery developed thus far, we can already prove the following for a general deformation problem.

Proposition 5.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a cohesive functor and $A \in X(\mathbb{F}_p)$ such that $T_{X_A} \simeq 0$. Then there exists a unique lift of A to a point in $\varprojlim_n X(\mathbb{Z}/p^n)$.*

Proof. Set $A_0 = A$, we inductively construct lifts against the tower of square zero extensions

$$\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{F}_p\,.$$

Suppose we have already constructed lifts A_k for $k \leq n$ for some n. Applying Proposition 3.20 inductively, we get that

$$T^{\mathbb{F}_p}_{X_{A_n}} \simeq T^{\mathbb{F}_p}_{X_{A_0}} \simeq 0.$$

Thus, since $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$ is a square zero extension with fiber $\mathbb{F}_p$, Proposition 4.7 implies that the fiber

$$X^{\mathbb{Z}/p^{n+1}}_{A_n} = \mathrm{fib}_{A_n}(X(\mathbb{Z}/p^{n+1}) \to \mathbb{Z}/p^n)$$

is contractible and we find an essentially unique lift A_{n+1}. This proves the claim. □

Of course, for an arbitrary functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ the natural map $X \to \varprojlim_n X(\mathbb{Z}/p^n)$ might not be an equivalence, meaning that in this generality we can only construct pro-p objects of X using this inductive method. In fact, we have that $\mathrm{cCAlg}_{\mathbb{Z}_p} \neq \varprojlim_n \mathrm{cCAlg}_{\mathbb{Z}/p^n}$. To remedy this problem we show that this limit admits a description via *p-complete* coalgebras. To do this, we first recall some facts about p-complete modules.

Definition 5.2. Let R be an $\mathbb{E}_\infty$-ring, then $M \in \mathrm{Mod}_R$ is called *p-complete* if the limit

$$\lim\left(\dots \xrightarrow{\cdot p} M \xrightarrow{\cdot p} M\right)$$

vanishes. We denote the full subcategory spanned by the p-complete modules by $(\mathrm{Mod}_R)^\wedge_p$.

Remark 5.3. The inclusion $(\mathrm{Mod}_R)^\wedge_p \hookrightarrow \mathrm{Mod}_\mathrm{R}$ admits a left adjoint which takes a module M to its *p-completion* given by the limit

$$\lim\left(\cdots \to M/p^2 \to M/p\right).$$

In fact, M is p-complete if and only if the natural map $M \to \lim M/p^n$ is an equivalence. This inherits a natural $R^\wedge_p$-module structure, thus p-completion also gives an equivalence of categories $(\mathrm{Mod}_R)^\wedge_p \simeq (\mathrm{Mod}_{R^\wedge_p})^\wedge_p$ which allows us to identify these in what follows.
The tensor product of p-complete modules is in general not p-complete. However, the category $(\mathrm{Mod}_R)^\wedge_p$ admits a symmetric monoidal structure given by the formula

$$M \otimes_{(\mathrm{Mod}_R)^\wedge_p} N := (M \otimes N)^\wedge_p.$$

With this monoidal structure the p-completion functor $\mathrm{Mod}_R \to (\mathrm{Mod}_R)^\wedge_p$ is strong monoidal, while the inclusion is only lax monoidal.

Definition 5.4. Let R be an $\mathbb{E}_\infty$-ring. We define the ∞-category of p-complete R-coalgebras is given by.

$$(\mathrm{cCAlg}_R)^\wedge_p := \mathrm{cCAlg}((\mathrm{Mod}_R)^\wedge_p).$$

Warning 5.5. Let R be a $\mathbb{E}_\infty$-ring. Notice that by our definition a p-complete R-coalgebra is the same as a p-complete $R^\wedge_p$-coalgebra and so we do not differentiate between the two notions. However, this is *not* the same as an $R_p^\wedge$-coalgebra whose underlying spectrum is p-complete. The process of p-completion does refine to a functor $\mathrm{cCAlg}_R \to (\mathrm{cCAlg}_{R_p^\wedge})_p^\wedge$, but it does not factor through the category $\mathrm{cCAlg}_{R_p^\wedge}$.

We now show check that the assignment $R \mapsto \mathrm{cCAlg}_R^{\mathrm{cn}}$ is subject to the machinery of deformation theory.

Lemma 5.6. *The following statements hold:*

(1) Suppose we have a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

such that the map $\pi_0 R \to \pi_0 S$ is surjective. Then the natural map

$$(\mathrm{cCAlg}_{R'}^{\mathrm{cn}})_p^\wedge \to (\mathrm{cCAlg}_R^{\mathrm{cn}})_p^\wedge \times_{(\mathrm{cCAlg}_S^{\mathrm{cn}})_p^\wedge} (\mathrm{cCAlg}_{S'}^{\mathrm{cn}})_p^\wedge$$

is an equivalence.

(2) For every connective $\mathbb{E}_\infty$-ring R, the natural map

$$(\mathrm{cCAlg}_R^{\mathrm{cn}})_p^\wedge \to \varprojlim_n (\mathrm{cCAlg}_{\tau_{\leq n} R}^{\mathrm{cn}})_p^\wedge$$

is an equivalence.

Proof. Ad 1.: Arguing as in the proof of Proposition 4.1, it suffices to show that the strong monoidal functor

$$(\mathrm{Mod}_{R'})_p^\wedge \to (\mathrm{Mod}_R)_p^\wedge \times_{(\mathrm{Mod}_S)_p^\wedge} (\mathrm{Mod}_{S'})_p^\wedge$$

is an equivalence. Indeed, given a point (M, N, h) in the pullback, the R'-module $M \times_{M \otimes_R S} N$ is again p-complete since p-completion commutes with limits. Thus, the inverse functor of Proposition 4.1 also induces a functor on the categories of p-complete modules. Moreover, we have that

$$((M \times_{M \otimes_R S} N) \otimes_{R'} R)_p^\wedge \simeq M_p^\wedge \simeq M$$
$$((M \times_{M \otimes_R} N) \otimes_{R'} S')_p^\wedge \simeq N_p^\wedge \simeq N,$$

where the first equivalences hold by Proposition 4.1, and the latter since M and N are to be p-complete. Finally, for $M \in (\mathrm{Mod}_{R'})_p^\wedge$, we compute that

$$(M \otimes_{R'} R)_p^\wedge \times_{(M \otimes_{R'} S)_p^\wedge} (M \otimes_{R'} S')_p^\wedge \simeq \left(M \otimes_{R'} R \times_{M \otimes_{R'} S} M \otimes_{R'} S'\right)_p^\wedge \simeq M_p^\wedge \simeq M,$$

where we have again used the result of Proposition 4.1 and the fact that p-completion commutes with limits.

Ad 2: This uses the exact same arguments applied to the equivalence of Corollary 4.5. □

Corollary 5.7. *For any $n \in \mathbb{N}$, the functor*

$$\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \qquad R \mapsto [(\mathrm{cCAlg}^{\mathrm{cn}}_R)^\wedge_p]^{\Delta^n}$$

is coherent and nilcomplete.

We now prove the crucial p-completeness result for $\mathbb{Z}_p$-modules. As before this will enable us to deduce the same result for coalgebras and allow us to tackle the actual problem of comparing coalgebras over $\mathbb{F}_p$, $\mathbb{Z}_p$ and $\mathbb{S}^\wedge_p$.

Proposition 5.8. *Let $\mathrm{Mod}^\wedge_{\mathbb{Z}_p} \subseteq \mathrm{Mod}_{\mathbb{Z}_p}$ denote the full subcategory spanned by the p-complete $\mathbb{Z}_p$-module spectra. Then the natural map*

$$\mathrm{Mod}_{\mathbb{Z}_p} \to \varprojlim_n \mathrm{Mod}_{\mathbb{Z}/p^n} \quad N \mapsto (N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n)$$

restricts to a strong monoidal equivalence

$$(\mathrm{Mod}_{\mathbb{Z}_p})^\wedge_p \simeq \varprojlim_n \mathrm{Mod}_{\mathbb{Z}/p^n}.$$

Proof. The functor admits a right adjoint which takes $(M_n) \in \varprojlim_n \mathrm{Mod}_{\mathbb{Z}/p^n}$ to the limit $\lim_n M_n$ taken in the category of $\mathbb{Z}_p$-modules. Since p-complete modules are closed under limits, the essential image of this functor is contained in $\mathrm{Mod}^\wedge_{\mathbb{Z}_p}$. Moreover, if $M \in \mathrm{Mod}^\wedge_{\mathbb{Z}_p}$, then we have that

$$\varprojlim_n (M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n) \simeq \varprojlim_n M/p^n \simeq M^\wedge_p \simeq M.$$

Hence, the counit of the adjunction is an equivalence on p-complete modules. Conversely, given $(N_k) \in \varprojlim_k \mathrm{Mod}_{\mathbb{Z}/p^k}$ write $N = \lim_k N$. We want to show that, for every n the natural map

$$N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \xrightarrow{\sim} N_n$$

is an equivalence. Since $N \otimes_{\mathbb{Z}_p} Z/p^n \simeq N/p^n$ and limits are exact, we have an equivalence

$$N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \simeq \lim_{k>n} (N_k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n).$$

Thus, the unit of the adjunction may be written as

$$\lim_{k>n} (N_k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n) \to \lim_{k>n} (N_k \otimes_{\mathbb{Z}/p^k} \mathbb{Z}/p^n) \simeq N_n$$

and so has fiber given by

$$F_n := \lim_{k>n} \left(N_k \otimes_{\mathbb{Z}/p^k} \mathrm{fib}(\mathbb{Z}/p^k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \to \mathbb{Z}/p^n) \right).$$

Now we compute the fiber of $\mathbb{Z}/p^k \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \to \mathbb{Z}/p^n$ as the module

$$\mathrm{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^k, \mathbb{Z}/p^n)[1] \simeq \mathbb{Z}/p^n[1].$$

The reduction map $\mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1}$ is induced by the map of projective resolutions

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\cdot p^k} & \mathbb{Z}_p \\ {\scriptstyle \cdot p}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}_p & \xrightarrow[\cdot p^{k-1}]{} & \mathbb{Z}_p, \end{array}$$

hence, on Tor it induces the multiplication by p map

$$\mathbb{Z}/p^n = \mathrm{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^k, \mathbb{Z}/p^n) \xrightarrow{\cdot p} \mathrm{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^{k-1}, \mathbb{Z}/p^n) = \mathbb{Z}/p^n.$$

Thus, if we have $k' > k > n$ such that $k' - k > n$, the transition map

$$F_{k'} = N_{k'} \otimes \mathrm{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^k, \mathbb{Z}/p^n) \to N_k \otimes \mathrm{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^{k-1}, \mathbb{Z}/p^n) = F_k$$

vanishes since the Tor-groups are p^n-torsion. Choosing a cofinal subset $S \subseteq \mathbb{N}_{>n}$ such that $|k' - k| > n$ for any distinct $k', k \in S$, we see that

$$\lim_{k>n} F_k \simeq \lim_{k\in S} F_k \simeq 0$$

vanishes. Thus, since limits are exact, the map $N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \xrightarrow{\sim} N_n$ is an equivalence.
To see that the functor $\mathrm{Mod}^{\wedge}_{\mathbb{Z}_p} \to \varprojlim_n \mathrm{Mod}_{\mathbb{Z}/p^n}$ is strong monoidal, we observe that since cofibers and limits are exact, we have for each n equivalences

$$\begin{aligned} (M \otimes_{\mathbb{Z}_p} N)^{\wedge}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n &\simeq \lim_k (M/p^k \otimes_{\mathbb{Z}_p} N/p^k)/p^n \\ &\simeq \lim_k \Big((M/p^n \otimes_{\mathbb{Z}_p} N/p^n) \otimes_{Z_p} \mathbb{Z}/p^k \Big) \\ &\simeq ((N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} (M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n))^{\wedge}_p. \end{aligned}$$

This proves the claim. □

Corollary 5.9. *We have an equivalence of categories*

$$(\mathrm{cCAlg}_{\mathbb{Z}_p})^{\wedge}_p \xrightarrow{\sim} \varprojlim_n \mathrm{cCAlg}_{\mathbb{Z}/p^n} \quad A \mapsto (A \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n)$$

with inverse taking a system of coalgebras (B_n) to the limit $\lim_n B_n$ taken in the category of (p-complete) $\mathbb{Z}_p$-modules, equipped with the induced p-complete $\mathbb{Z}_p$-coalgebra structure.

Proof. This follows from Proposition 5.8, arguing as in the proof of Proposition 4.1. □

Corollary 5.10. *Let $X(-) = (\mathrm{cCAlg}^{\mathrm{cn}}_{-})^{\Delta^0}$ and $A \in X(\mathbb{F}_p)$ such that $T_{X_A} \simeq 0$. Then the space of lifts of A to a p-complete $\mathbb{Z}_p$-coalgebra is contractible*

Proof. Combine Proposition 5.1 and Corollary 5.9. □

Corollary 5.11. *Let $\varphi : B \to A$ be a map of connective, formally étale $\mathbb{F}_p$-coalgebras. Then the space of lifts of φ to a map of p-complete $\mathbb{Z}_p$-coalgebras $B' \to A'$ is contractible.*

Proof. Let $\mathcal{X}(-) = \mathrm{cCAlg}^{\mathrm{cn}}_{-}$. By Proposition 4.19 the natural map

$$T_{\mathcal{X}^{\Delta^1}_{\varphi}} \to T_{\mathcal{X}^{\Delta^0}_{B}}$$

is an equivalence, but since B is formally étale we have $T_{\mathcal{X}^{\Delta^0}_{B}} \simeq 0$. Hence, the claim follows by applying Proposition 5.1 to the functor $\mathcal{X}^{\Delta^1}$ and using Corollary 5.9. □

Having shown this, we can now construct a functor which is analogous to the classical Witt-Vectors, which allow us to pass from étale $\mathbb{F}_p$-algebras to $\mathbb{Z}_p$-algebras.

Theorem 5.12. *Let $\mathcal{C} \subseteq (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{Z}_p})^{\wedge}_p$ denote the full subcategory spanned by those coalgebras A for which $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is formally étale. Then the base change functor*

$$\mathcal{C} \to \mathrm{cCAlg}^{\mathrm{cn,f\acute{e}t}}_{\mathbb{F}_p} \qquad A \mapsto A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$$

is fully faithful and essentially surjective. In particular, the quasi inverse defines a functor

$$W_p : \mathrm{cCAlg}^{\mathrm{cn,f\acute{e}t}}_{\mathbb{F}_p} \to (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{Z}_p})^{\wedge}_p$$

which is fully faithful and satisfies $W_p(A) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \simeq A$ for every connective, formally étale $\mathbb{F}_p$-coalgebra A.

Proof. Combine Corollary 5.10 and Corollary 5.11. □

We now turn our attention to the leap from $\mathbb{Z}_p$ to $\mathbb{S}^{\wedge}_p$. The following proposition shows that, for an arbitrary cohesive and nilcomplete functor, a $\mathbb{Z}_p$-valued point which has vanishing $\mathbb{F}_p$-tangent complex admits a unique lift to a $\mathbb{S}^{\wedge}_p$-valued point. This is surprising, as we do not actually require any information about the $\mathbb{Z}_p$-tangent complex, everything is determined by what happens modulo p.

Proposition 5.13. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a cohesive and nilcomplete functor and let $A \in X(\mathbb{Z}_p)$ such that $T_{X_{A \otimes_{\mathbb{Z}_p} \mathbb{F}_p}} \simeq 0$. Then A admits an essentially unique lift to $X(\mathbb{S}^{\wedge}_p)$.*

Proof. We inductively construct lifts against the Postnikov Tower

$$\cdots \to \tau_{\leq 2}\mathbb{S}^{\wedge}_p \to \tau_{\leq 1}\mathbb{S}^{\wedge}_p \to \tau_{\leq 0}\mathbb{S}^{\wedge}_p \simeq \mathbb{Z}_p.$$

Write $A = A_0$, $S_n = \tau_{\leq n}\mathbb{S}^{\wedge}_p$, $M_n = \pi_n S_n$ and assume we have already constructed a unique lift A_n to $X(S_n)$. Consider the square zero extension

$$M_{n+1}[n+1] \to S_{n+1} \to S_n.$$

Since $M_{n+1} = \pi_{n+1}S_{n+1}$ is concentrated in a single degree, the S_n-action factors through $S_0 = \mathbb{Z}_p$. Moreover, since $\pi_{n+1}S_{n+1}$ is of finite p-torsion, the action further factors through $\mathbb{Z}/p^k$ for some $k \geq 0$. Thus, Proposition 3.20 implies that we have an equivalence

$$T^{M_{n+1}[n+1]}_{X_{A_n}} \simeq \Sigma^n T^{M_{n+1}}_{X_{A_n}} \simeq T^{M_{n+1}}_{X_{A_n \otimes_{S_n} \mathbb{Z}/p^k}}.$$

Arguing as in Proposition 3.21 with respect to the square zero extension

$$\mathbb{F}_p \to \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1},$$

we see that we have a cofiber sequence

$$T^{M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{F}_p}_{X_{A_n\otimes_{S_n}\mathbb{Z}/p^{k-1}}} \to T^{M_{n+1}}_{X_{A_n\otimes_{S_n}\mathbb{Z}/p^k}} \to T^{M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{Z}/p^{k-1}}_{X_{A_n\otimes_{S_n}\mathbb{Z}/p^{k-1}}}.$$

For the left hand term, Proposition 3.20 gives the equivalence

$$T^{M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{F}_p}_{X_{A_n\otimes_{S_n}\mathbb{Z}/p^{k-1}}} \simeq T^{M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{F}_p}_{X_{A\otimes_{\mathbb{Z}_p}\mathbb{F}_p}} \simeq T_{X_{A\otimes_{\mathbb{Z}_p}\mathbb{F}_p}} \otimes_{\mathbb{F}_p} (M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{F}_p) \simeq 0,$$

where we have used that, since M_{n+1} is finitely generated, the $\mathbb{F}_p$-module $M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{F}_p$ is perfect. For the right hand term we replace M_{n+1} with $M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{Z}/p^{k-1}$ and repeat the argument, inductively yielding equivalences

$$T^{M_{n+1}}_{X_{A_n\otimes_{S_n}\mathbb{Z}/p^k}} \simeq T^{M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{Z}/p^{k-1}}_{X_{A_{n-1}\otimes_{S_{n-1}}\mathbb{Z}/p^{k-1}}} \simeq \cdots \simeq T^{M_{n+1}\otimes_{\mathbb{Z}/p^k}\mathbb{F}_p}_{X_{A\otimes_{\mathbb{Z}_p}\mathbb{F}_p}} \simeq 0.$$

In total, this shows that $T^{M_{n+1}[n+1]}_{X_{A_n}} \simeq 0$, and hence A_n admits an essentially unique lift to $X(S_{n+1})$. Thus, the fiber over A of the map

$$X(\mathbb{S}_p^\wedge) \simeq \varprojlim_n X(S_n) \to X(\mathbb{Z}_p)$$

is contractible and we are done. □

Lemma 5.14. *Write $\mathcal{X}(-) = \mathrm{cCAlg}^{\mathrm{cn}}_-$ and $\mathcal{Y}(-) = (\mathrm{cCAlg}^{\mathrm{cn}}_-)^\wedge_p$. Then the p-completion map $f : \mathcal{X} \to \mathcal{X}'$ induces an equivalence*

$$T^M_{(\mathcal{X}^{\Delta^n})_\xi} \to T^M_{(\mathcal{Y}^{\Delta^n})_{f(\xi)}}$$

for every $\mathbb{F}_p$-module M, $n \in \mathbb{N}$ and $\xi \in \mathcal{X}(\mathbb{F}_p)^{\Delta^n}$.

Proof. For any $\mathbb{F}_p$-algebra R the p-completion map gives an equivalence $\mathrm{Mod}_R \xrightarrow{\sim} (\mathrm{Mod}_R)^\wedge_p$, since multiplication by some power of p is nullhomotopic over $\mathbb{F}_p$. In particular, this applies to the split square zero extension $\mathbb{F}_p \oplus M$ for any $M \in \mathrm{Mod}_{\mathbb{F}_p}$ and so the natural map $\mathcal{X}(\mathbb{F}_p \oplus M) \to \mathcal{Y}(\mathbb{F}_p \oplus M)$ is an equivalence as well. Consequently, we also obtain natural equivalences between the fibers

$$(\mathcal{X}^{\Delta^n})^{\mathbb{F}_p\oplus M}_\xi \to (\mathcal{Y}^{\Delta^n})^{\mathbb{F}_p\oplus M}_{f(\xi)},$$

which induces the equivalence of spectra

$$T^M_{(\mathcal{X}^{\Delta^n})_\xi} \to T^M_{(\mathcal{Y}^{\Delta^n})_{f(\xi)}}$$

as claimed. □

Corollary 5.15. *Let $X(-) = (\mathrm{cCAlg}^{\mathrm{cn}}_-)^{\Delta^0}$ and $A \in X(\mathbb{F}_p)$ such that $T_{X_A} \simeq 0$, then the space of lifts of A to a p-complete $\mathbb{S}_p^\wedge$-coalgebra is contractible.*

Proof. Write $Y(-) = ((\mathrm{cCAlg}^{\mathrm{cn}}_-)^\wedge_p)^{\Delta^0}$. Then by Lemma 5.14 we have an equivalence $T_{X_A} \simeq T_{Y_A} \simeq 0$. Hence, we can apply Proposition 5.10 to obtain an essentially unique lift $A' \in Y(Z_p)$. Further applying Proposition 5.13 to A' yields our claim. □

Thus, we can pointwise lift $\mathbb{F}_p$-coalgebras with vanishing tangent complex to $\mathbb{S}_p^\wedge$. If we moreover consider *formally étale coalgebras*, we can make this lifting functorial in a coalgebraic analogue of the *Spherical Witt Vectors* construction for $\mathbb{E}_\infty$-algebras over $\mathbb{F}_p$.

Corollary 5.16. *Let $\varphi : B \to A$ be a map of $\mathbb{F}_p$-coalgebras such that A and B are formally étale. Then the space of lifts of φ to a map $\varphi' : B' \to A'$ of p-complete $\mathbb{S}_p^\wedge$-coalgebras is contractible.*

Proof. Let $\mathcal{X}(-) = \mathrm{cCAlg}^{\mathrm{cn}}_{-}$ and $\mathcal{Y}(-) = (\mathrm{cCAlg}^{\mathrm{cn}}_{-})^\wedge_p$. By Proposition 5.11 the map φ admits an essentially unique lift to a point $\psi \in \mathcal{Y}(\mathbb{Z}_p)^{\Delta^1}$. Moreover, Lemma 5.14 yields an equivalence $T_{\mathcal{X}^{\Delta^1}_\varphi} \simeq T_{\mathcal{Y}^{\Delta^1}_\varphi}$. Since both A and B are formally étale Proposition 4.19 gives equivalences

$$T_{\mathcal{X}^{\Delta^1}_\varphi} \xrightarrow{\sim} T_{\mathcal{X}^{\Delta^0}_B} \simeq 0$$

Hence, we can apply Proposition 5.13 to the functor $\mathcal{Y}^{\Delta^1}$ and the point $\psi \in \mathcal{Y}^{\Delta^1}$, proving the claim. □

Theorem 5.17. *Denote by $\mathcal{C} \subseteq (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$ the full subcategory spanned by those coalgebras A such that $A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$ is formally étale. Then the functor*

$$\mathcal{C} \to \mathrm{cCAlg}^{\mathrm{cn,f\acute{e}t}}_{\mathbb{F}_p} \qquad A \mapsto A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$$

is fully faithful and essentially surjective. In particular, the quasi inverse defines a functor

$$W_{\mathbb{S}_p^\wedge} : \mathrm{cCAlg}^{\mathrm{cn,f\acute{e}t}}_{\mathbb{F}_p} \to (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$$

which is fully faithful and satisfies $W_{\mathbb{S}_p^\wedge}(A) \otimes_{\mathbb{S}_p} \mathbb{F}_p \simeq A$ for every connective, formally étale $\mathbb{F}_p$-coalgebra A.

Proof. Combine Corollary 5.15 and Corollary 5.16. □

5.2. **Homology coalgebras.** As observed in Example 2.14, for every space X and every $\mathbb{E}_\infty$-ring R, the R-homology $R[X]$ carries a natural R-coalgebra structure, which is a stronger invariant than its underlying R-module. We now want to apply our results and see what can be said about the deformation theoretic behavior of homology coalgebras. To do this, we first need to compute the cotangent complex of the $\mathbb{F}_p$-cohomology.

Definition 5.18. A space $X \in \mathcal{S}$ is called p-finite if the following conditions hold:

1. (1) The space X is truncated.
2. (2) The set $\pi_0 X$ is finite.
3. (3) For each $n \geq 1$ and $x \in X$, we have that $\pi_n(X, x)$ is a finite p-group.

We denote the full subcategory of $\mathcal{S}$ spanned by the p-finite spaces as $\mathcal{S}_p$ and call $\mathcal{S}_p^\vee =: \mathrm{Pro}(\mathcal{S}_p)$ the category of *p-profinite* spaces.

Remark 5.19. We can regard $\mathcal{S}_p^\vee$ as the category of "formal limits" of p-finite spaces $\varprojlim X_\alpha$. As such there is a functor $\mathcal{S}_p^\vee \to \mathcal{S}$ which takes a formal limit to the actual limit in $\mathcal{S}$. This functor admits a left adjoint given by $Y \mapsto \varprojlim_{Y_\alpha \to Y} Y_\alpha$, where the limit runs over all maps from a p-finite space Y_α to Y.

Lemma 5.20. *Let X be a space and $\varprojlim X_\alpha$ be its p-profinite completion. Then the natural map of cohomology rings*

$$\varinjlim \mathbb{F}_p^{X_\alpha} \to \mathbb{F}_p^X$$

is an equivalence.

Proof. This is immediate since the Eilenberg-MacLane spaces $K(\mathbb{F}_p, n)$ are p-finite. □

Proposition 5.21. *[Mandell, Lurie] Let X be a space, then the $\mathbb{F}_p$-cohomology $\mathbb{F}_p^X$ is a formally étale $\mathbb{F}_p$-algebra.*

Proof. Since the functor $R \mapsto L_{R/\mathbb{F}_p}$ commutes with colimits, the claim follows from the fact that $L_{\mathbb{F}_p^X/\mathbb{F}_p} \simeq 0$ for every p-finite space X which is proven in [Lur11, Proposition 2.4.12]. □

Thus we obtain the following result about the homology coalgebra of a finite space X with coefficients in a connective $\mathbb{F}_p$-algebra R:

Corollary 5.22. *Let X be a finite space and R be an $\mathbb{F}_p$-algebra, then $R[X]$ is a formally étale R-coalgebra.*

Proof. From Proposition 5.21 we get that

$$L_{R^X/R} \simeq L_{\mathbb{F}_p^X/\mathbb{F}_p} \otimes_{\mathbb{F}_p} R \simeq 0.$$

Since X is finite, the coalgebra $R[X]$ is dualizable with dual given by R^X, so the claim follows from Proposition 4.13. □

Moreover, for the case $R = \mathbb{F}_p$, we can use Theorem 5.17 to give a partial answer to our initial question about lifts of the coalgebra $\mathbb{F}_p[X]$.

Corollary 5.23. *Let X be a finite space, then $\mathbb{F}_p[X]$ admits a unique lift to a p-complete $\mathbb{S}_p^\wedge$-coalgebra given by $W_{\mathbb{S}_p^\wedge}(\mathbb{F}_p[X]) \simeq (\mathbb{S}[X])_p^\wedge$. Moreover, for any other finite space Y the natural map*

$$\mathrm{Map}_{(\mathrm{cCAlg}_{\mathbb{S}_p^\wedge})_p^\wedge}((\mathbb{S}[Y])_p^\wedge, (\mathbb{S}[X])_p^\wedge) \to \mathrm{Map}_{\mathrm{cCAlg}_{\mathbb{F}_p}}(\mathbb{F}_p[Y], \mathbb{F}_p[X])$$

is a homotopy equivalence.

Proof. Combine Corollary 5.22 and Theorem 5.17. □

6. Where to go from here

We finish our discussion by explaining some of the shortcomings of our results and sketch a possible way to proceed towards a coalgebraic analogue of Mandell's Theorem. The first missing puzzle piece is the cotangent complex of a coalgebra A, which we have been unable to give a solid definition of. The second and more important one is the relation to the *coalgebra Frobenius*. We conjecture that the class of *perfect* coalgebras defined via this map give examples of non-dualizable formally étale coalgebras. In particular, this conjecture would imply that the $\mathbb{F}_p$-homology of *any* space X is formally étale.

6.1. **The cotangent complex of a coalgebra.** One of the first questions that arose during this project turned out to be one of the most subtle and tricky ones, namely:

Question 6.1. *What is the cotangent complex of a coalgebra A?*

Clearly, the existence of a single spectrum controlling the deformation theory of A would be immensely useful. However, it is not immediately clear what the universal property of such a spectrum should be, i.e. which space of derivations it should (co)represent. Some inspiration can be gleamed from Proposition 4.12. There we had seen that, for $\varphi : B \to A$ a map of R-coalgebras with A dualizable and M an R-module, we have an equivalence

$$\mathrm{Der}_\varphi(B, C_A(M)) \simeq \mathrm{Map}_{A^\vee}(L_{A^\vee/R}, \varphi_*^\vee \mathrm{map}_R(B, M)).$$

To get rid of the dependence on the second coalgebra B one is tempted to take $B = R$ such that $\mathrm{map}_R(B, M) \simeq M$. However, not every coalgebra A admits a map $R \to A$, much less a canonical one. The only natural choice for a map that is not the initial map would yield the following:

Definition 6.2 (Preliminary 1.)**.** Let R be an $\mathbb{E}_\infty$-ring and $A \in \mathrm{cCAlg}_R$. The cotangent complex of A, if it exists, is the R-module L_A corepresenting the functor

$$\mathrm{Mod}_R \to \mathrm{Mod}_R \qquad M \mapsto \mathrm{der}_{\mathrm{id}}(A, C_A(M))$$

There are however several problems with this. Firstly, it is entirely unclear from the definition whether L_A vanishing would actually imply A being formally étale. Moreover, in the dualizable case it would lead to the rather awkward formula

$$L_A \simeq L_{A^\vee/R} \otimes_{A^\vee} A.$$

Although somewhat plausible, this again gives us little information about what can actually be deduced in the case that $L_A \simeq 0$. This leaves us with several options, lest we accept that there is no good notion of one singular cotangent complex. For one we could work with *coaugmented* coalgebras, namely coalgebras together with a map $R \to A$. For the purpose of understanding homology coalgebras this would correspond to considering pointed spaces instead of just spaces, an entirely acceptable compromise, but beyond the scope of this paper.

A different approach would be to give up on the idea of corepresentability and instead hope for a colimit preserving functor. For example, the functor

$$\mathrm{Mod}_R \to \mathrm{Mod}_R \qquad M \mapsto C_A(M) := \mathrm{cofib}(A \xrightarrow{\varepsilon} \Omega_A^\infty M).$$

seems to have no chance of preserving limits, but since colimits of coalgebras are formed underlying, colimits are not out of the race. This leads us to the following idea:

Definition 6.3 (Preliminary 2). Let R be an $\mathbb{E}_\infty$-ring and $A \in \mathrm{cCAlg}_R$. We say that A admits a cotangent complex $L_A := C_A(R)$ if the functor $C_A(-) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ commutes with colimits. In this case we have $C_A(M) \simeq L_A \otimes M$ for every $M \in \mathrm{Mod}_R$

This definition is highly speculative, as the only coalgebras we know to admit a cotangent complex in this sense are the formally étale coalgebras, for which the functor $C_-(A)$ is constant. Conversely, if A admits a cotangent complex then L_A vanishes if and only if A is formally étale. Hence, the spectrum L_A is precisely the obstruction to A being formally étale, which is the kind of conceptual clarity we are looking for. While we lose any direct comparison to the cotangent complex of $A^\vee$ this is not entirely surprising, since the property of being formally étale is defined very differently for $A^\vee$. Overall, we are left with the following dream:

Conjecture 6.4. *Let R be an $\mathbb{E}_\infty$-ring. Then any $A \in \mathrm{cCAlg}_R$ admits a cotangent complex in the sense of Definition 6.3.*

Whether this precise conjecture turns out to be true or not, the takeaway should be that the modules $C_A(M)$ are the obstruction towards A being formally étale. Moreover, while the functor $A \mapsto C_A(M)$ is very complicated, the dependence on M should be relatively tame. That is, for fixed A it should be possible to describe the functor $M \mapsto C_A(M)$ in terms of a formula involving $C_A(R)$. However, because $C_A(M)$ no longer has a direct relation to any space of derivations or tangent complex, we cannot leverage results like Proposition 3.18 to obtain such a formula. We understand this as an indication that for these questions, the formalism may have reached its limit.

6.2. **The Frobenius.** The most lacking thing about our results is the class of coalgebras that we can currently apply them to. As of now, we are unable to give examples of formally étale coalgebras which are not dualizable. In particular, we cannot describe the deformation theory of $R[X]$ for spaces X which are not finite. Attempts to reduce to the dualizable case all seem to fail for the following reason: Even though we may write $X = \varinjlim_i X_i$ where each X_i is finite, giving the formula $R[X] = \varinjlim_i R[X_i]$, there is no reason why the functor $\Omega^\infty_-(M) : \mathrm{cCAlg}_R \to \mathrm{cCAlg}_R$ should commute with colimits. Indeed, write $f_M : R \to R \oplus M$ for inclusion, then by definition $\Omega^\infty_-(M) = f_{M,!} f_M^*$. The functor f_M^* commutes with colimits, and from Proposition 2.5 and the converse of the adjoint functor theorem we can deduce that $f_{M,!}$ commutes with κ-filtered colimits for some regular cardinal κ. Thus, the class of formally étale coalgebras is closed under κ-filtered colimits, but κ is, in general, not countable. This goes to show that the deformation theory of non-dualizable coalgebras is richer and more interesting than that of the Ind-completion of dualizable coalgebras and requires additional input. One contender for this additional input is the *Coalgebra Frobenius* constructed by Nikolaus:

Theorem 6.5 (Nikolaus)**.** *Let $\mathcal{C} = (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$, then there exists a natural transformation $\psi_p : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$ which on an object $A \in \mathcal{C}$ is given by the composition*

$$\psi_p : A \xrightarrow{\Delta_A^{\otimes p}} (A^{\otimes p})^{hC_p} \xrightarrow{\mathrm{can}} (A^{\otimes p})^{tC_p} \xrightarrow{\sim} A,$$

where the final map is the inverse of the Tate Diagonal, see [NS17, Theorem III.1.7].

Given this map, we are naturally led to define *perfect* coalgebras as follows:

Definition 6.6. We say that $A \in (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$ is *perfect* if the coalgebra Frobenius $\psi_p : A \to A$ is a homotopy equivalence. We denote the full subcategory spanned by the perfect coalgebras by $(\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^{\wedge,\mathrm{perf}}_p \subseteq (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$.

Example 6.7. Let X be any space. Then $(\mathbb{S}[X])^\wedge_p$ is a perfect coalgebra since we have that

$$\mathbb{S}[X]^\wedge_p \simeq (\mathbb{S}^\wedge_p[\operatorname*{colim}_X *])^\wedge_p \simeq (\operatorname*{colim}_X \mathbb{S}^\wedge_p)^\wedge_p.$$

On $\mathbb{S}^\wedge_p$ the map ψ_p is necessarily given by the identity, because $\mathbb{S}^\wedge_p$ is the terminal p-complete $\mathbb{S}^\wedge_p$-coalgebra. Thus, by naturality ψ_p is given by the identity on $(\mathbb{S}[X])^\wedge_p$ as well.

We conjecture that this Frobenius map is related to the deformation theory of coalgebras in a similar way to the Algebra Frobenius, in that it provides a sufficient condition for a coalgebra to be formally étale.

Conjecture 6.8. *Let $A \in (\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^\wedge_p$, then for any $M \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p}$, the coalgebra Frobenius $\psi_p : A \to A$ induces the zero map on the R-module $C_{A\otimes_{\mathbb{S}_p^\wedge}\mathbb{F}_p}(M) = \mathrm{cofib}(A \xrightarrow{\eta_A} \Omega^\infty_A(M))$.*

Corollary 6.9. *If Conjecture 6.8 holds, then the base change functor*

$$(\mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{S}_p^\wedge})^{\wedge,\mathrm{perf}}_p \to \mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{F}_p} \qquad A \mapsto A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$$

is fully faithful and factors through the full subcategory $\mathrm{cCAlg}^{\mathrm{cn,fét}}_{\mathbb{F}_p} \subseteq \mathrm{cCAlg}^{\mathrm{cn}}_{\mathbb{F}_p}$.

Proof. Since $\psi_p : A \xrightarrow{\sim} A$ is an equivalence it induces an equivalence on $A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$ and thus on $C_{A\otimes_{\mathbb{S}_p^\wedge}\mathbb{F}_p}(M)$ as well. However, since it also induces the zero map on the latter we get that $C_{A\otimes_{\mathbb{S}_p^\wedge}\mathbb{F}_p}(M) \simeq 0$. Thus, $A \otimes_{\mathbb{S}_p^\wedge} \mathbb{F}_p$ is formally étale and the claim follows from Theorem 5.17. □

Combining this with Example 6.7 would allow us to fully answer our initial question about homology coalgebras.

Corollary 6.10. *If Conjecture 6.8 holds, then for any space X the $\mathbb{F}_p$-chains $\mathbb{F}_p[X]$ are formally étale. In particular $\mathbb{F}_p[X]$ admits a unique and functorial lift to a p-complete $\mathbb{S}^\wedge_p$-coalgebra given by $\mathbb{S}^\wedge_p[X] = W_{\mathbb{S}_p^\wedge}(\mathbb{F}_p[X])$.*

The fact that Conjecture 6.8 needs to be checked for every connective $\mathbb{F}_p$-module should be understood as an extension of our failure to find a cotangent complex. Indeed, if $\mathbb{F}_p[X]$

admits a cotangent complex in the sense of Definition 6.3, then to obtain Corollary 6.10 it would suffice to show that ψ_p induces the zero map on $C_{A\otimes_{\mathbb{S}_p^\wedge}\mathbb{F}_p}(\mathbb{F}_p) = L_{A\otimes_{\mathbb{S}_p^\wedge}\mathbb{F}_p}$. However, even for this specific module the conjecture is difficult to attack from our present position. The problem is the tricky right adjoint $\mathrm{cCAlg}_{\mathbb{F}_p\oplus\mathbb{F}_p} \to \mathrm{cCAlg}_{\mathbb{F}_p}$ appearing in the definition of $C_{A\otimes_{\mathbb{S}_p^\wedge}\mathbb{F}_p}(\mathbb{F}_p)$. Because there is no known formula for this functor, attempts to verify the conjecture have thus far been unsuccessful in all non-trivial cases. This warrants further investigation of the coalgebra Frobenius in general and Conjecture 6.4 as well as Conjecture 6.10 in specific.

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