Lord of the $K(n)$-local rings: The two Towers

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This talk is an overview of the the preprint [BSSW23] by Barthel, Schlank, Stapleton and Weinstein. The authors compute the rational homotopy groups of the $K(n)$-local sphere by showing that the chromatic vanishing conjecture holds rationally.

1 The Setup

Fix a prime p. Our goal in life is to understand the homotopy groups of the sphere spectrum $\pi_*\mathbb{S}$. We can do this "one prime at a time" by considering the localization $\mathbb{S}_{(p)}$ where we invert all primes $\neq p$. This is still too hard to attack directly. Here chromatic homotopy theory provides an iterative approach by approximating the p-local sphere via stages of the *chromatic tower*

$$\mathbb{S}_{(p)} \simeq \lim\left(\cdots \to L_2\mathbb{S} \to L_1\mathbb{S} \to L_0\mathbb{S} \simeq \mathbb{Q}\right).$$

Here $L_n = L_{E(n)}$ is the Bousfield localization at the Morava E-theory spectrum of height n. The localization L_n is meant to capture information at "height $\leq n$". We can describe the associated graded of this tower via the *chromatic fracture square*

$$\begin{array}{ccc} L_n\mathbb{S} & \longrightarrow & L_{K(n)}\mathbb{S} \\ \downarrow & & \downarrow \\ L_{n-1}\mathbb{S} & \longrightarrow & L_{n-1}L_{K(n)}\mathbb{S} \end{array}$$

where $K(n)$ is the n-th Morava K-theory spectrum. The localization $L_{K(nm)}$ is supposed to capture information "at height $= n$". Thus, if we could understand the spectrum $L_{K(n)}\mathbb{S}$ we would have a shot at working our way up the chromatic tower.

Question 1.1. How do we access $\pi_*L_{K(n)}\mathbb{S}$?

The idea is to descend along the unit map $L_{K(n)}\mathbb{S} \to E$, where E denotes the Morava E-theory spectrum of height n. Let us collect some facts about E-theory to make this precise.

Theorem 1.2 (Goerss-Hopkins-Miller, Lurie)**.** *Let $\int_{\mathrm{perf}} \mathrm{Mfg}^{=n}$ denote the the category of pairs (k, Γ) where k is a perfect $\mathbb{F}_p$-algebra and Γ is a formal group of height n over k. Then there exists a fully faithful functor*

$$\int_{\mathrm{perf}} \mathrm{Mfg}^{=n} \to \mathrm{CAlg}(\mathrm{Sp}_{K(n)}) \quad (k,\Gamma) \mapsto E(k,\Gamma)$$

together with a canonical isomorphism $\mathrm{Spf}(\pi_0 E(k,\Gamma)) \simeq \mathrm{Def}_\Gamma$. Here Def_Γ denotes the deformation functor which assigns to any infinitesimal thickening $k' \to k$ the set of deformations of Γ to k' i.e.

$$\mathrm{Def}_\Gamma(k') = \pi_0\mathrm{fib}_\Gamma\left(\mathrm{Mfg}(k') \to \mathrm{Mfg}(k)\right).$$

Moreover, there exists a (non-canonical) isomorphism

$$\pi_* E(k,\Gamma) \simeq W(k)[[u_1,\dots u_{n-1}]][\beta^{\pm}] \quad |\beta| = 2.$$

The spectrum $E(k,\Gamma)$ is called Morava E-theory or Lubin-Tate-theory. Over an algebraically closed field of characteristic p, all formal groups non-canonically are isomorphic and thus, we can once and for all fix an algebraic closure k of $\mathbb{F}_p$ and formal group of height n over k. Then $E := E(k,\Gamma)$ is what is commonly referred to as E-theory when people are lazy about notation and we will also follow this convention.

Definition 1.3. The Morava stabilizer group $\mathbb{G}$ is defined as

$$\mathbb{G} := \mathrm{Aut}(\Gamma, k),$$

i.e. the group of pairs of autormorphisms (σ, f) fitting into a commutative diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\;f\;} & \Gamma \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\;\sigma\;} & \mathrm{Spec}(k) \end{array}$$

This is a profinite group sitting in a split exact sequence

$$\mathrm{Aut}(\Gamma) \to \mathbb{G} \to \mathrm{Gal}(k/\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$$

i.e. we have $\mathbb{G} \simeq \mathrm{Aut}(\Gamma) \rtimes \mathrm{Gal}(k/\mathbb{F}_p)$ where the elements of the Galois group act on Γ by applying them to the coeffecients of the power series.

Since $E(-,-)$ is a functor, we immediately get an action of $\mathbb{G}$ on E via maps of $K(n)$-local ring spectra. Moreover, this action is continuous with respect to the profinite topology on $\mathbb{G}$ and a certain adic topology on E. More precisely, it can be promoted to a continuous G-action on the spectral stack $\mathrm{Spf}(E)$. The following theorem should be thought of as saying that the unit map $L_{K(n)}\mathbb{S} \to E$ is a pro-Galois cover with Galois group $\mathbb{G}$. This precise formulation of the theorem is due to Gregoric.

Theorem 1.4 (Devinatz-Hopkins)**.** *The unit $L_{K(n)}\mathbb{S} \to E$ is equivariant with respect to the trivial $\mathbb{G}$-action on $L_{K(n)}\mathbb{S}$ and induces an equivalence*

$$L_{K(n)}\mathbb{S} \simeq \mathbb{E}^{h\mathbb{G}} := \mathcal{O}(\mathrm{Spf}(E)/\mathbb{G})$$

Analyzing the associated descent datum gives a spectral sequence

$$H^*_{\mathrm{cont}}(\mathbb{G}, \pi_* E) \Rightarrow \pi_* L_{K(n)}\mathbb{S}$$

which is often called the *Devinatz-Hopkins spectral sequence.* This means all we need to do is understand the action of the Morava stabilizer group on $\pi_* E$ and we can start computing. This is however notoriously difficult, since we are trying to understand a Galois action and an action of power series via composition simultaneously. The paper [BSSW23] shows that, if we rationalize the spectral sequence completely collapses and we obtain isomorphisms

$$H^*_{\mathrm{cont}}(\mathbb{G}, \pi_* E) \otimes \mathbb{Q}_p \simeq H^*_{\mathrm{cont}}(\mathbb{G}, \pi_0 E) \otimes \mathbb{Q}_p \simeq H^*_{\mathrm{cont}}(\mathbb{G}, W(k)) \otimes \mathbb{Q}_p \simeq \Lambda_{\mathbb{Q}_p}(\zeta_1, \dots \zeta_n)$$

Where $\mathbb{G}$ acts on $W(k)$ via its quotient $\mathrm{Gal}(\mathbb{F}_p)$. None of these equivalences are clear and we will discuss them in detail. For now, this yields the main theorem of the paper.

Theorem 1.5 ([BSSW23]). *There is an isomorphism of graded $\mathbb{Q}$-algebras*

$$\pi_* L_{K(n)}\mathbb{S} \otimes \mathbb{Q} \simeq \Lambda_{\mathbb{Q}_p}(\zeta_1, \dots \zeta_n)$$

with $\deg(\zeta_i) = 1 - 2i$.

The first step of this computation is the equivalence

$$H^*_{\text{cont}}(\mathbb{G}, \pi_* E) \otimes \mathbb{Q}_p \simeq H^*_{\text{cont}}(\mathbb{G}, \pi_0 E) \otimes \mathbb{Q}_p,$$

which is crucial as we can use the geometric interpretation of $\mathrm{Spf}(\pi_0 E)$ to understand its cohomology. Let us begin with this before diving into geometry.
For this, we use the following fact: Suppose G is a profinite group acting continuously on an abelian group A and suppose we have a closed normal subgroup $H \subseteq G$. Then $H^*_{\text{cont}}(H, A) = 0$ implies that $H^*_{\text{cont}}(G, A) = 0$. This is a concequence of the Hochschild-Lyndon-Serre spectral sequence, or, if you are condensed enough, the fact that derived functors compose,i.e.

$$A^{hG} \simeq (A^{hH})^{hG/H}$$

Proposition 1.6. *For every $t \neq 0$ we have that*

$$H^*_{\text{cont}}(\mathbb{G}, \pi_t E) \otimes \mathbb{Q} \simeq H^*_{\text{cont}}(\mathbb{G}, \pi_t E \otimes \mathbb{Q}) \simeq 0$$

Proof. We have an exact sequence

$$\mathrm{Aut}(\Gamma) \to \mathbb{G} \to \mathrm{Gal}(k/\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}.$$

Moreover, the assignment $n \mapsto [n] \in \mathrm{End}(\Gamma)$ extends to an injection $\mathbb{Z}_p \to \mathrm{End}(\Gamma)$ and hence we have a normal subgroup $\mathbb{Z}_p^\times \subseteq \mathrm{Aut}(\Gamma)$. Thinking of $(\mathbb{Z}_p, +)$ as a subgroup of $\mathbb{Z}_p^\times, \cdot)$ via $1 \mapsto 1 + p$, we obtain another exact sequence

$$\mathbb{Z}_p \to \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Gamma)/\mathbb{Z}_p.$$

Thus, it suffices to show that $H^*_{\text{cont}}(\mathbb{Z}_p, \pi_t E \otimes \mathbb{Q}) = 0$ where, tracing the definitions, the generator of $\mathbb{Z}_p$ acts via multiplication by $(1+p)^t$. This cohomology is computed by the complex $\mathbb{Q}_p$-modules

$$\pi_t E \otimes \mathbb{Q} \xrightarrow{(1+p)^t - 1} \pi_t E \otimes \mathbb{Q}$$

which is clearly acyclic since we assumed $t \neq 0$. ☐

The next step is the main computation of the paper. Write $W(k) = W$ and $A = \pi_0 E$.

Theorem 1.7 ([BSSW23]). *For every i, the inclusion $W \subseteq A$ induces a split injection*

$$H^i_{\text{cont}}(\mathbb{G}, W) \to H^i_{\text{cont}}(\mathbb{G}, A)$$

whose cokernel is killed by a power of p independent of i.

Remark 1.8. In fact, for height $n = 1, 2$ the cokernel is known to completely vanish. The *chromatic vanishing conjecture* states that this is true for all heights. Whether this is true is wide open.

The splitting of Theorem 1.7 already exists on the level of $\mathbb{G}$-representations.

Proposition 1.9. *The map $W \to A$ admits a $\mathbb{G}$-equivariant retraction, giving a G-equivariant equivalencd $A \simeq W \oplus A^c$.*

Proof. Previous Talk. ☐

As promised, the way this theorem is attacked is using the geometry of Lubin-Tate space $LT := \mathrm{Spf}(A)$. The idea is that $H^*_{\mathrm{cont}}(\mathbb{G}, A)$ should be thought of as the cohomology of he quotient stack $LT/\mathbb{G}$. After passing to the rigid analytic fiber (i.e. rationalizing in a sense), this space can also be described as a quotient of the *Drinfeld upper half plane* $\mathcal{H}$, which is given by $\mathbb{P}^{n-1}_{\mathbb{Q}_p}$ with some points removed, together with the natural action of $\mathrm{GL}_n(\mathbb{Q}_p)$. The cohomology of $\mathcal{H}/\mathrm{GL}_n$ we can actually compute, and this results in the theorem. To make this story precise, we need some technology.

2 Technology

2.1 Rigid analytic geometry

In the following let $K = W[1/p]$. What we want is a version of algebraic geometry in which the generic fiber $A\widehat{\otimes}K \simeq K[[x_1, \ldots, x_{n-1}]]$ cohomologically behaves like a p-adic n-dimensional formal disk. This is provided by the theory of *adic spaces*.

Definition 2.1. A *Huber ring* is a topological ring A such that there exists an open subring $A_0 \subseteq A$ and a finitely generated ideal $I \subseteq A_0$ such that $\{I^n\}_{n\geq 0}$ is a neighbourhood basis of 0. A *Huber pair* (A, A^+) consists of a Huber ring A and an open, integrally closed subring $A^+ \subseteq A$ such that for each $f \in A^+$ and each $n \geq 0$, there exists a large $N > 0$ such that $I^N f^k \in I^n$ for all k.

Example 2.2.
1. The primordial example is the pair $(\mathbb{Q}_p, \mathbb{Z}_p)$ with the p-adic topology. Notice that here the Huber ring A is actually a field, so we really need the ideal I to come from an open subring. Moreover, $(\mathbb{Z}_p, \mathbb{Z}_p)$ is also a Huber pair, while $(\mathbb{Q}_p, \mathbb{Q}_p)$ is *not*.

2. The pair $(W[[x_1, \ldots, x_n]], W[[x_1, \ldots, x_n]])$ is a Huber pair with respect to the $(p, x_1, \ldots x_n)$-adic topology.

Definition 2.3. Let (A, A^+) be a Huber pair. The *adic spectrum* $\mathrm{Spa}(A, A^+)$ is defined to be the set of isomorphism classes of continuous multiplicative functions $|-| : A \to H \cup \{0\}$ such that $|A^+| \leq 1$, with H a totally ordered abelian group. We equip $\mathrm{Spa}(A, A^+)$ with the topology generated by the open subsets

$$U\left(\frac{f_1, \ldots, f_m}{g}\right) := \{|-| \in \mathrm{Spa}(A, A^+) \mid |f_i| \leq |g|\}$$

Example 2.4.
1. The space $\mathrm{Spa}(\mathbb{Q}_p[[t]], \mathbb{Z}_p[[t]])$, i.e. the generic fiber of $\mathrm{Spa}(\mathbb{Z}_p[[t]], \mathbb{Z}_p[[t]])$ is a closed disk of radius one. To see this, observe that for every $\alpha \in \mathbb{Q}_p$ with $|\alpha| \leq 1$ we have a map $\mathbb{Q}_p[[t]] \to \mathbb{Z}$ given by taking f to $|f(\alpha)|$. In fact, one can show that all continouous valuations arise in this way.

2. Similarly, the adic spectrum of $(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ is the open disk of radius one. Here the angled brackets indicate the ring of power series which converge in the p-adic topology.

Definition 2.5. Let (A, A^+) be a Huber pair and $X = \mathrm{Spa}(A, A^+)$. We define sheaves $\mathcal{O}^+_X \subseteq \mathcal{O}_X$ on X as follows. On a generic open $U \subseteq X$ as above, we set $\mathcal{O}_X(U)$ as the completion of $A[f_i/g]$ and $\mathcal{O}^+_X(U)$ as the integral closurse in $\mathcal{O}_X(U)$ of $A^+[f_i/g]$. In general this only defines a presheaf which we can sheafifiy, but for the spaces which appear here it will already be a sheaf. An *adic space* is a triple $(Y, \mathcal{O}_Y, \mathcal{O}^+_Y)$ which is locally isomorphic to the adic spectrum of a Huber pair, which is called an *affinoid adic space*. A *rigid analytic space* is an adic space which is locally isomorhpic to $\mathrm{Spa}(A, A^+)$, where A is a quotient of $K\langle T_1, \ldots, T_m\rangle$ and A^+ is the set of power-bounded elements in A.

There is a good theory of étale cohomology for adic spaces. An étale map of adic spaces is roughly a map which is locally glued together from maps of Huber pairs $(A, A^+) \to (B, B^+)$, such that $A \to B$ is étale and B^+ is the integral closure of A^+ is in B.

Definition 2.6. Let X be a rigid analytic space. The pro-étale site of X is defined as the ringed site whose objects are given by formal limits $U = \varprojlim_i U_i$ of rigid analytic spaces $U_i \to X$ which are étale over X. Coverings are given by maps which are jointly surjective on underlying topological spaces. The integral structure sheaf is defined as the completion

$$\hat{\mathcal{O}}_X^+(U) := \varprojlim_i (\mathcal{O}_X(U_i))_p^\wedge$$

and the structure sheaf is $\hat{\mathcal{O}}_X := \hat{\mathcal{O}}_X^+[1/p]$.

The étale cohomology of a rigid analytic space X should now be the derived global sections of the sheaf $\hat{\mathcal{O}}_X^+$. However, it is unclear what we mean by that. Since we just went through all these definitions to consider carry along the topology and completeness on this sheaf, simply disregrarding it and taking the global sections in $D(\mathbb{Z}_p)$ is clearly foolish. However, topological abelian groups do not even form an abelian category and have no immediate derived analogue. Since we are dealing with *adic* topologies however, recent develepoments of Clausen-Scholze give us a good world to work in via the notions of condensed and solid abelian groups.

2.2 Condensed Mathematics

We give the briefest introduction to the condensed world.

Definition 2.7. The pro-étale site of a point $*_{\text{proét}}$ is the category of profinite sets with finite families of jointly surjective continuous maps as covers. For any category $\mathcal{C}$ we define

$$\text{Cond}(\mathcal{C}) := \text{Sh}^{\text{hyp,acc}}(*_{\text{proét}}, \mathcal{C}),$$

as the category of hypercomplete, accessible $\mathcal{C}$-valued sheaves on $*_{\text{proét}}$[1]. For $\mathcal{C}$ being topological spaces/sets/groups etc. there is a functor

$$\mathcal{C} \to \text{Cond}(\mathcal{C}) \quad X \mapsto (S \mapsto \underline{X}(S) := \text{Map}_{\text{cont}}(S, X))$$

which is fully faithful on compactly generated topological spaces.

Since these are categories of sheaves, good properties/strucutre of $\mathcal{C}$ gives the same on $\text{Cond}(\mathcal{C})$. For example, Cond(Ab) is abelian and $D(\text{Cond(Ab)}) \simeq \text{Cond}(D(\mathbb{Z}))$. There is a notion of completeness in condensed Abelian groups which is called *solid*.

Definition 2.8. Let $\underline{\mathbb{Z}[-]} : \text{Cond(Set)} \to \text{Cond(Ab)}$ be the free condensed abelian group and $S = \lim_i S_i$ be a profinite set. The free solid Abelian group on $\underline{S}$ is defined to be

$$\mathbb{Z}^{\blacksquare}[S] := \varprojlim \underline{\mathbb{Z}[S_i]}$$

A solid abelian group is a condensed abelian group A such that for each profinite set S, every map $\underline{\mathbb{Z}[S]} \to A$ factors uniquely through $\mathbb{Z}^{\blacksquare}[S]$.

The full category Solid $\subseteq$ Cond(Ab) is again abelian. The point is that our cohomology theories on rigid analytic spaces will take values in $D(\text{Solid})$.

[1]If $\mathcal{C}$ is a 1-category, then all sheaves are automatically hypercomplete and accessible

Lemma 2.9. *Let M be a topological abelian group which is separated and complete for a linear topology. Moreover, let G be a profinite group acting continuously on M. Then $\underline{M}$ is a solid abelian group with a $\underline{G}$-action.*

This applies to all the groups we are considering. Moreover, continuous cohomology groups now admit an interpretation as derived fixed points. Let G be a solid abelian group and denote by Solid_G the category of solid abelian groups with a G-action. This is also an abelian category and the functor $\mathrm{Solid}_G \to \mathrm{Solid}$ which takes G-fixed points admits a derived analogue

$$(-)^{hG} : D(\mathrm{Solid})_G \to D(\mathrm{Solid})$$

Proposition 2.10. *If we are in the situation of Lemma 2.9 there is an equivalence*

$$\underline{H^*_{\mathrm{cont}}(G, A)} \simeq \pi_{-*}\underline{A}^{h\underline{G}}.$$

Back to geometry, for any rigid analytic space X, there is a natural map

$$\pi : X_{\text{proét}} \to *_{\text{proét}},$$

and we define

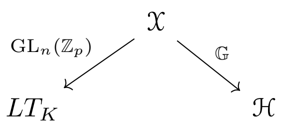
$$\Gamma^+_{\mathrm{cond}}(X_{\text{proét}}) := \pi_* \hat{\mathcal{O}}^+_X \in \mathrm{D(Solid)}$$

to be the derived pushforward of the integral structure sheaf.

3 The geometric input, aka the two towers

Theorem 3.1. *[Faltings, Scholze-Weinstein] There exists a perfectoid space $\mathcal{X}$ with $\mathbb{G}$-action and a $\mathbb{G}$-equvariant pro-étale $\mathrm{Gl}_n(\mathbb{Z}_p)$-covering $\mathcal{X} \to LT_K$. Moreover, the quotient space $\mathcal{X}/\mathbb{G}$ is given by the rigid-analytic space $\mathcal{H} := \mathbb{P}^{n-1}_{\mathbb{Q}_p} \setminus \bigcup_H H$ where H runs over the $\mathbb{Q}_p$-hyperplanes in $\mathbb{P}^{n-1}$. The residual $\mathrm{Gl}_n(\mathbb{Z}_p)$-action is induced from the natural action on projective space.*

The space $\mathcal{H}$ is called the *Drinfeld upper half plane* and should be considered an p-adic analogue of $\mathbb{P}^{n-1}_{\mathbb{R}} \setminus \mathbb{R}$. The theorem shows that we have a diagram of adic spaces



Since $\mathcal{X}$ is perfectoid and the maps are pro-étale covers, one can show that they induce equivalences

$$\underline{R\Gamma}(LT_{K,\text{proét}}, \hat{\mathcal{O}^+})^{h\mathbb{G}} \simeq \underline{R\Gamma}(\mathcal{X}, \hat{\mathcal{O}}^+_{\mathcal{X}})^{h\mathbb{G}\times h\mathrm{GL}_n(\mathbb{Z}_p)} \simeq \underline{R\Gamma}(\mathcal{H}, \hat{\mathcal{O}}^+_{\mathcal{H}})^{h\mathrm{GL}_n(\mathbb{Z}_p)} \in D(\mathrm{Solid}),$$

coming from equivalences of the associated analytic quotient stacks. Thus, on the analytic generic fiber, the mysterious action of $\mathbb{G}$ on Lubin Tate space can be interchanged with the action of $\mathrm{GL}_n(\mathbb{Z}_p)$ on the Drinfeld space $\mathcal{H}$. Before we explain how to levarage this to prove the main theorem, let us explain some of the workings of Theorem 3.1. The cover $\mathcal{X} \to LT_K$ is constructed as follows: Recall that LT parametrizes deformations of the formal group Γ. For each m we can define a variant of Lubin-Tate space by asking for deformations with an m-level strucutre, which is roughly a trivialization of the p^m-torsion of the deformations. Denote the this space of deformations with m-level structure as LT^m, then $\mathrm{GL}_n(\mathbb{Z}_p)$ acts on the choice of coordinartes for the p^m-torsions via $\mathrm{GL}_n(\mathbb{Z}/p^m)$. The space $\mathcal{X}$ is then defined as the limit of the generic fibers

$$\mathcal{X} = \varprojlim(\cdots \to LT_K^2 \to LT_K^1 \to LT_K)$$

which is also called the *Lubin-Tate tower*. On the other side, it turns out that $\mathcal{H}_K$ is the generic fiber of a formal scheme parametrizing of deformations of

$$M = \Gamma \oplus \Gamma^{(p)} \oplus \cdots \oplus \Gamma^{(p^{n-1})}$$

where $\Gamma^{(p^k)}$ denotes the pullback of Γ along the p^k-Frobenius on $\overline{\mathbb{F}}_p$. The map $\mathcal{H}_K \to \mathcal{H}$ is a $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ torsor, and taking the limit along a tower of level structures on $\mathcal{H}_K$ the deformation problems become "equivalent" in the limit by a theorem of Scholze-Weinstein. This gives a $\mathrm{Aut}(\Gamma)$-torsor $\mathcal{X} \to \mathcal{H}_K$ and hence the composition $\mathcal{X} \to \mathcal{H}_K \to \mathcal{H}$ is a $\mathbb{G}$-torsor as claimed.
To leverage this theorem, we need to understand what the derived global sections have to do with the group cohomology groups we were thinking about initially.

Theorem 3.2 (BSSW)**.** *We can approximate the equivariant cohomology groups as follows.*

1. *There exists a $\mathbb{G}$-equivariant morhpism of derived Solid W-algebras*

$$A[\varepsilon] \to \underline{R\Gamma}(LT_{K,\text{proét}}, \hat{\mathcal{O}^+})$$

2. *There exists a $\mathrm{GL}_n(\mathbb{Z}_p)$-equivaraint map of derived solid $\mathbb{Z}_p$-algebras*

$$\mathbb{Z}_p[\varepsilon] \to \underline{R\Gamma}(\mathcal{H}, \hat{\mathcal{O}}^+)$$

For each of these maps, the cohomology groups of the cofiber are annhiliated by a single power of p across all degrees.

Equipped with this theorem, we can now prove Theorem 1.7. Applying homotopy fixed points, we obtain two maps

$$A[\varepsilon]^{h\mathbb{G}} \to \underline{R\Gamma}(LT_{K,\text{proét}}, \hat{\mathcal{O}^+})^{h\mathbb{G}} \cong \underline{R\Gamma}(\mathcal{H}, \hat{\mathcal{O}}^+)^{h\mathrm{GL}_n(\mathbb{Z}_p)} \leftarrow \mathbb{Z}_p[\varepsilon]^{h\mathrm{GL}_n(\mathbb{Z}_p)}.$$

Taking homology and inverting p, we obtain an isomorphism

$$H^*_{\text{cts}}(\mathbb{G}, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p[\varepsilon] \cong H^*_{\text{cts}}(\mathrm{GL}_n(\mathbb{Z}_p), \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[\varepsilon]$$

The equivariant splitting $A \simeq W \oplus A^c$ gives an isomorphism of the left-hand site with

$$\left(H^*_{\text{cts}}(\mathbb{G}, K) \oplus H^*_{\text{cts}}(\mathbb{G}, A^c) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right) \otimes \mathbb{Q}_p[\varepsilon]$$

The groups $\mathrm{GL}_n(\mathbb{Z}_p)$ and $\mathbb{G}$ are both p-adic Lie-groups whose Lie-algebras become isomorphic after base changing to an algebraic closure. Thus, by a theorem of Lazard and a descent argument we can compute that

$$H^*_{\text{cts}}(\mathrm{GL}_n(\mathbb{Z}_p), \mathbb{Q}_p) \cong H^*_{\text{cts}}(\mathbb{G}, K) \cong H^*(\mathfrak{gl}_n(\mathbb{Q}), \mathbb{Q}) \otimes \mathbb{Q}_p \cong \Lambda_{\mathbb{Q}_p}(x_1, \ldots, x_{2n-1}).$$

Thus, comparing dimensions of $\mathbb{Q}_p$-vector spaces yields that $H^*_{\text{cts}}(\mathbb{G}, A^c) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$ as claimed.

References

[BSSW23] Tobias Barthel, Tomer Schlank, Nathaniel Stapleton, and Jared Weinstein. *On the rationalization of the K(n)-local sphere.* 2023. URL: `https://math.bu.edu/people/jsweinst/chromatic.pdf`.